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INDUCED TOPOLOGICAL PROPERTIES

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## TABLE OF CONTENTS

	Page
ACKNOWLEDGMENTS . . . . .	ii
Chapter	
I.    INTRODUCTION . . . . .	1
II.   SUBSPACES. . . . .	4
III.  FUNCTIONS. . . . .	21
IV.   DECOMPOSITION SPACES . . . . .	27
V.    PRODUCT SPACES . . . . .	38
VI.   TOPOLOGICAL GROUPS . . . . .	44
INDEX . . . . .	55
LITERATURE CITED. . . . .	58

## CHAPTER I

### INTRODUCTION

When working in topology, one often deals with various types of subspaces, decomposition spaces, and product spaces. It is useful to know which topological properties are inherited by these spaces from the original space and what additional restrictions are sufficient to ensure that certain properties are "induced."

The word "induced" is meant to be flexible. It is not the purpose here to define rigorously a new concept of induced properties. In Chapter II the induced properties are those inherited by subspaces from the original space. On the other hand, in Chapter V the properties of a family of spaces which are possessed by the product space and also the properties of the product space possessed by each coordinate space are considered "induced" properties.

This paper consists of ninety-four theorems involving many types of induced properties. In order to make these properties readily accessible to the reader, an index of properties has been included.

Chapter II, entitled "Subspaces," deals with the properties encountered in a first year course in general topology which are inherited by various subspaces. Some of the properties treated here are first and second countability, separation properties, compactness,  $\sigma$ -locally finite bases, paracompactness, uniformities, uniform continuity, pseudo-metrizability, uniform covers, completeness, nets, and totally boundedness.

The third chapter investigates the properties of functions on a topological space  $X$  possessed by their restrictions to certain subspaces of  $X$ . Examples of the functions treated here are: continuous functions, open and closed functions, compact functions, quasi-compact functions, quasi-open functions, monotone functions, quasi-monotone functions and retractions. Properties inherited by the image spaces of functions are not investigated here. This is the topic for another thesis [2].

If in addition to being a topological space,  $X$  is a group in which the operations are "continuous" then  $X$  has special inheritance properties. These properties are investigated in Chapter VI. It is shown, for instance, that all topological groups are regular and that certain local properties are induced in the entire group.

The notation and definitions of this paper are greatly influenced by J. L. Kelley's General Topology. The convention of using the square ■ to indicate the end of a proof is used here. Square brackets, e.g. [2], are used to denote the number of the reference to the bibliography at the end. The intersection of the sets  $A$  and  $B$  is denoted multiplicatively by  $AB$  except in Chapter VI in which  $AB$  denotes the product in a group of the sets  $A$  and  $B$ . Set intersection there is indicated by  $\cap$ . The notation  $\{x: - - -\}$  is read "the set of all  $x$  such that - - -."

The development of decomposition spaces and product spaces, although approached in several different ways in the references, has been unified by means of the projection map  $P$ .

Since various forms of definitions appear in the literature for the concepts used in this thesis, many definitions are included as those concepts are encountered.

In this thesis a topological space will consist of a set  $X$  and a family  $T$  of subsets of  $X$  such that  $T$  is closed under the formation of finite intersections and arbitrary unions, and  $X$  is a member of  $T$ . The members of  $T$  will be referred to as open sets in the topological space  $X$ .



## CHAPTER II

## SUBSPACES

The induced topological properties in this chapter are the topological properties inherited by the various subspaces of a topological space with the relative topology. The questions investigated here are: "What properties are inherited by all subspaces?" and "What restrictions are necessary on the subspaces for certain properties to be inherited?" The properties examined are in general the ones encountered in a first year course in general topology.

Theorem 2.1 If  $X$  satisfies the first axiom of countability then each subspace  $A$  of  $X$  satisfies the first axiom of countability.

Proof Let  $x$  be in  $A$ . Let  $B = \{B_1, B_2, \dots\}$  be a base of the neighborhood system of  $x$ . Consider the collection

$$C = \{AB_1, AB_2, \dots\}.$$

Let  $V$  be an open set in  $A$  containing  $x$ . It follows that  $V$  will be in the form  $AU$  for some open set  $U$  in  $X$ . Since  $B$  is a base, there is some  $B_k$  in  $B$  so that  $x$  is in  $B_k$  and  $B_k \subset U$ . Observe that  $AB_k \subset V$  and hence each neighborhood of  $x$  in  $A$  contains a member of  $C$  which in turn contains  $x$ . ■

Theorem 2.2 If  $X$  satisfies the second axiom of countability then each subspace  $A$  of  $X$  satisfies the second axiom of countability.

The proof is similar to the preceding one and is omitted.

Separability is not necessarily inherited by subspaces as is illustrated by the following example.

Example 2.1 Let  $X$  consist of the points of the cartesian plane with non-negative  $y$ -coordinates. Let the basis for the topology of  $X$  consist of all open discs contained in  $X$  whose boundary does not intersect the  $x$ -axis, together with all open discs contained in  $X$  which are tangent to the  $x$ -axis, including the point of tangency.

This well known example is Hausdorff, regular, and first countable. The space  $X$  is separable since the collection of points in  $X$  with rational coordinates is dense in  $X$ . The  $x$ -axis with the relative topology is discrete and therefore not separable.

Theorem 2.3 If the subsets  $B$  and  $C$  are separated in  $X$  and  $A$  is a subspace of  $X$  such that  $BA \neq \emptyset$  and  $CA \neq \emptyset$ , then  $BA$  and  $CA$  are separated in  $A$ .

Proof An equivalent property to that of being separated is that  $B$  and  $C$  are both open and closed in  $B \cup C$ . It follows that  $BA$  and  $CA$  are both open and closed in  $A(B \cup C)$ , and hence  $BA$  and  $CA$  are both open and closed in  $BA \cup CA$ . Thus  $BA$  and  $CA$  are separated in  $A$ . ■

Corollary Each subspace of a totally disconnected topological space is totally disconnected.

Corollary If  $A$  is a subspace of  $X$  and  $C$  is a connected subset of  $A$ , then  $C$  is a connected subset of  $X$ .

Theorem 2.4 Let  $A$  be a subspace of  $X$ . If  $B$  and  $C$  is a separated pair of subsets of  $A$ , then  $B$  and  $C$  are separated in  $X$ .

Proof Both  $B$  and  $C$  are open and closed in  $B \cup C$  considered as a

subspace of A. The relative topology of  $B \cup C$  in A is exactly the same as the relative topology of  $B \cup C$  in X, hence both B and C are open and closed in X. ■

Theorem 2.5 If X is  $T_0$ ,  $T_1$ , or  $T_2$  then each subspace of X is  $T_0$ ,  $T_1$ , or  $T_2$  respectively.

Proof Suppose X is  $T_2$ . Let A be a subspace of X and let x and y be members of A such that  $y \neq x$ . There are open disjoint sets U, V in X such that x is in U and y is in V. The required disjoint neighborhoods of x and y in A are  $UA$  and  $VA$  respectively. The proofs for  $T_0$  and  $T_1$  are similar.

Theorem 2.6 If X is normal then every closed subset A of X is normal.

Proof Let B and C be closed disjoint subsets in A. Since A is closed in X it follows that B and C are closed in X. There are open disjoint subsets U and V in X such that  $B \subset U$  and  $C \subset V$ . The desired disjoint neighborhoods in A containing B and C are  $UA$  and  $VA$  respectively. ■

To show that the theorem is false without the condition of A being closed, consider the following example.

Example 2.2 Let X consist of the four points a, b, c, and d. Let the open sets be  $\{a\}$ ,  $\{a, b\}$ ,  $\{a, c\}$ ,  $\{a, b, c\}$ , X and  $\emptyset$ . The closed subsets are  $\{b, c, d\}$ ,  $\{c, d\}$ ,  $\{b, d\}$ ,  $\{d\}$ , X and  $\emptyset$ . There are no non-empty disjoint closed sets so that the normality of X follows trivially. Let  $A = \{a, b, c\}$ . The closed subsets  $\{b\}$ , and  $\{c\}$  are disjoint. Each open set contains a, therefore there are no disjoint neighborhoods.

Definition 2.1 A topological space X is completely normal if and only if X is  $T_1$  and for each pair of separated sets H, K there is a dis-

joint pair of open sets  $U, V$  such that  $U$  contains  $H$  and  $V$  contains  $K$ .

Theorem 2.7 If  $X$  is completely normal then each subspace of  $X$  is completely normal.

Proof Let  $A$  be a subspace of  $X$  and let  $H, K$  be a separated pair of sets in  $A$ . By Theorem 2.4 it follows that  $H, K$  is a separated pair of sets in  $X$ . There are open sets  $U$  and  $V$  in  $X$  which are disjoint and such that  $H \subset U$  and  $K \subset V$ . The sets  $UA$  and  $VA$  are disjoint open sets containing  $H$  and  $K$  respectively. The property of  $T_1$  is inherited by Theorem 2.5. ■

Theorem 2.8 If  $X$  is regular then each subspace of  $X$  is regular.

Proof Let  $A$  be a subspace of  $X$ . Let  $x$  be a member of  $A$ , let  $U$  be an open set in  $A$  containing  $x$ . There is an open set  $V$  in  $X$  such that  $U = VA$ . Since  $X$  is regular there exists a closed neighborhood  $W$  of  $x$  in  $X$  such that  $W \subset V$ . Now  $WA$  is a closed neighborhood of  $x$  in  $A$  such that  $WA \subset U$ . ■

Definition 2.2 A topological space  $X$  is completely regular if and only if for each  $x$  in  $X$  and for each neighborhood  $U$  of  $x$  there is a continuous function  $f$  on  $X$  into the interval  $[0,1]$  (with the usual topology) such that  $f(x) = 0$  and  $f$  is identically one on the complement of  $U$ . If, in addition,  $X$  is  $T_1$  then  $X$  is a Tychonoff space.

Theorem 2.9 If  $X$  is completely regular then each subspace of  $X$  is completely regular.

Proof Let  $x$  be in the subspace  $A$  of  $X$  and let  $U$  be an open set in  $A$  which contains  $x$ . There is an open set  $V$  in  $X$  such that  $U = VA$ . Since  $X$  is completely regular, there exists a continuous function  $f$  on  $X$  into  $[0,1]$  such that  $f(x) = 0$  and  $f$  is identically one on  $X-V$ . The restriction  $f/A$  is continuous (see Theorem 3.1) and has the properties required in the

definition. ■

Corollary Each subspace of a Tychonoff space is a Tychonoff space.

Definition 2.3 A topological space  $X$  is locally connected if and only if for each point  $x$  in  $X$  and each neighborhood  $U$  of  $x$  the component of  $U$  containing  $x$  is in turn a neighborhood of  $x$ .

Theorem 2.10 Let  $X$  be a locally connected space, then each open subset of  $X$  is locally connected.

Proof Let  $A$  be an open subset of  $X$ , let  $x$  be in  $A$ , and let  $V$  be an open set in  $A$  containing  $x$ . There is an open set  $U$  in  $X$  such that  $V = AU$ . Now  $UA$  is open in  $X$  so that the component  $C$  of  $UA$  containing  $x$  is a neighborhood of  $x$  in  $X$  and therefore is a neighborhood of  $x$  in  $A$ . ■

Definition 2.4 A topological space is connected im kleinen if and only if for each  $x$  in  $X$  and for each open set  $U$  containing  $x$  there is an open set  $V$  containing  $x$  such that  $V \subset U$  and for each  $y$  in  $V$  there is a connected subset of  $U$  containing both  $x$  and  $y$ .

Theorem 2.11 If  $X$  is connected im kleinen then each open subset is connected im kleinen.

Proof Let  $A$  be an open subset of  $X$ , let  $x$  be in  $A$  and let  $U$  be an open set in  $A$  containing  $x$ . The set  $U$  is also open in  $X$  since  $A$  is open in  $X$  and therefore there exists an open set  $V$  with the properties described in the definition. ■

Definition 2.5 A topological space  $X$  is compact if and only if each open covering of  $X$  has a finite subcover which covers  $X$ .

Definition 2.6 A topological space  $X$  is locally compact if and only if each  $x$  in  $X$  has a compact neighborhood.

Theorem 2.12 Both compactness and locally compactness are inherited

by closed subspaces of  $X$ .

Proof Suppose  $X$  is locally compact. Let  $A$  be a closed subset of  $X$ , and let  $x$  be in  $A$ . Since  $X$  is locally compact, there is a compact neighborhood  $V$  of  $x$  in  $X$ . The intersection  $AV$  is a closed subset of  $V$  and is therefore compact and a neighborhood of  $x$  in  $A$ . The fact that closed subsets of compact sets are compact is stated in most topology texts. ■

Theorem 2.13 If the locally compact topological space  $X$  is regular then each open subset  $A$  is locally compact.

Proof Let  $x$  be a member of  $A$ . There is a compact neighborhood  $V$  of  $x$  in  $X$ . There is an open set  $U$  contained in  $V$  such that  $x$  is in  $U$  and by the regularity of  $X$  there is a closed neighborhood  $W$  of  $x$  contained in  $U$ . The set  $W$  is a closed subset of the compact set  $V$  and is therefore compact. ■

Theorem 2.14 Let  $A$  be a subspace of the topological space  $X$  and let  $C$  be a subset of  $A$ . The set  $C$  is compact in  $A$  if and only if  $C$  is compact in  $X$ .

Proof Let  $\{U_a\}$  be an open covering of  $C$  in  $A$ . Each  $U_a$  is of the form  $V_a \cap A$ , where each  $V_a$  is open in  $X$ . Now  $\{V_a\}$  covers  $C$  in  $X$  and therefore there is a finite subcovering

$$\{V_{a_1}, \dots, V_{a_n}\}$$

which also covers  $C$ . The corresponding elements of  $\{U_a\}$

$$\{U_{a_1}, \dots, U_{a_n}\}$$

form the required subcover.

Conversely, suppose  $C$  is compact in  $A$ . Let  $\{V_a\}$  be an open cover of  $C$  in  $X$ . Let  $U_a = V_a \cap A$  for each  $V_a$  in  $\{V_a\}$ . The collection  $\{U_a\}$  is an open covering of  $C$  in  $A$  and hence there is a finite subcover

$$\{U_{a_1}, \dots, U_{a_n}\}$$

which covers  $C$  in  $A$ . The elements  $V_{a_1}, \dots, V_{a_n}$  cover  $C$ . ■

Definition 2.7 A topological space  $X$  is sequentially compact if and only if each sequence of elements of  $X$  has a convergent subsequence.

Theorem 2.15 If  $X$  is a sequentially compact topological space then each closed subset is sequentially compact.

Proof Let  $A$  be a closed subset of  $X$ . Let  $\{x_n\}$  be an infinite sequence in  $A$ . Since  $\{x_n\}$  is also an infinite sequence in  $X$ , there is a convergent subsequence  $\{x_{n_k}\}$  which converges to some  $x$  in  $X$ . However,  $x$  is a limit of the subsequence and hence must belong to  $A$ . It follows that the subsequence  $\{x_{n_k}\}$  is a convergent subsequence in  $A$ . ■

Definition 2.8 A topological space  $X$  is countably compact if and only if every infinite subset of  $X$  has a limit point in  $X$ .

Theorem 2.16 Each closed subset of a countably compact space  $X$  is countably compact.

The proof is similar to that of Theorem 2.15.

Definition 2.9 A family  $A$  of subsets of a topological space  $X$  is locally finite if and only if for each  $x$  in  $X$  there is a neighborhood  $U$  of  $x$  which intersects only finitely many of the members of  $A$ .

Definition 2.10 A family  $A$  of subsets of a topological space is  $\sigma$ -locally finite if and only if  $A$  is a countable union of locally finite families.

Definition 2.11 A family  $A$  of subsets of a topological space  $X$  is discrete if and only if for each  $x$  in  $X$  there is a neighborhood  $U$  of  $x$  which intersects only one member of  $A$ .

Definition 2.12 A family  $A$  of subsets of a topological space is  $\sigma$ -discrete if and only if  $A$  is the countable union of discrete families.

Theorem 2.17 If  $X$  has a  $\sigma$ -locally finite ( $\sigma$ -discrete) base then each subset  $B$  has a  $\sigma$ -locally finite ( $\sigma$ -discrete) base.

Proof Let  $A$  be a  $\sigma$ -locally finite base for  $X$ . The collection  $A$  is of the form  $\bigcup_{i=1}^{\infty} U_i$ , where each  $U_i$  is locally finite. The family

$$S = \left\{ uB : u \in A \right\}$$

is a base for  $B$  and furthermore  $S = \bigcup_{i=1}^{\infty} v_i$  where

$$v_i = \left\{ uB : u \in U_i \right\}.$$

Let  $i$  be a positive integer and let  $x$  be in  $B$ . There is an open set  $W$  containing  $x$  and which intersects only finitely many of the members of  $U_i$  say  $u_1, \dots, u_n$ . Thus  $WB$  intersects only  $u_1B, \dots, u_nB$  or in other words only finitely many of the elements of  $v_i$ . It follows that  $S$  is a  $\sigma$ -locally finite base for  $B$ . The proof for the  $\sigma$ -discrete case is similar. ■



Recall that normality is not in general inherited by subspaces. Theorem 2.17 together with the following lemma yields a condition for  $X$  in which all subspaces are normal.

Lemma A regular topological space  $X$  which has a  $\sigma$ -locally finite base is normal.

For a proof see [3], p. 127.

Regularity and the property of having a  $\sigma$ -locally finite base are inherited by all subspaces (Theorem 2.8 and Theorem 2.17). The next theorem follows from the last remark and the above lemma.

Theorem 2.18 Normality is inherited by all subspaces of a regular topological space with a  $\sigma$ -locally finite base.

Definition 2.13 A cover  $B$  of a set  $X$  is a refinement of a cover  $C$  of  $X$  if and only if each member of  $B$  is a subset of at least one member of  $C$ .

Definition 2.14 A topological space  $X$  is paracompact if and only if it is regular and each open cover of  $X$  has a locally finite refinement which is an open cover.

Theorem 2.19 If  $X$  is a paracompact topological space then each closed subset is paracompact.

Proof Regularity is inherited by Theorem 2.8. Let  $A$  be a closed subset of  $X$  and let  $\{V_a\}$  be an open cover of  $A$ . Each  $V_a$  is of the form  $AU_a$  where each  $U_a$  is open in  $X$ . The collection

$$\{X-A\} \cup \{U_a\}$$

is an open cover for  $X$  and therefore there is an open locally finite re-

finement  $\{W_b\}$ . The required locally finite open refinement of  $\{V_a\}$  is  $\{AW_b\}$ . ■

In order to investigate properties of uniform convergence, uniform continuity, and Cauchy sequences without introducing a metric, the concept of a uniform topological space is useful. The uniform space, according to J. L. Kelly, is due to A. Weil [3].

Definition 2.15 Let  $U$  be a subset of the cartesian product  $X \times X$ . The inverse  $U'$  of  $U$  is the subset of  $X \times X$  consisting of all  $(x, y)$  such that  $(y, x)$  is in  $U$ .

Definition 2.16 Let  $U$  and  $V$  be subsets of  $X \times X$ . The symbol  $U \cdot V$  denotes the subset of  $X \times X$  consisting of all  $(x, y)$  such that there is a  $z$  in  $X$  so that  $(x, z)$  is in  $V$  and  $(z, y)$  is in  $U$ .

Definition 2.17 Let  $X$  be a set and let  $U$  be a collection of non-empty subsets of  $X \times X$ . The collection  $U$  is a uniformity for  $X$  if and only if

- (1) each member of  $U$  contains the diagonal;
- (2)  $u'$  is in  $U$  for each  $u$  in  $U$ ;
- (3) for each  $u$  in  $U$  there is a  $v$  in  $U$  such that  $v \cdot v \subset u$ ;
- (4)  $U$  is closed under finite intersections;
- (5) if  $u \in U$  and  $u \subset v \subset X \times X$ , then  $v \in U$ .

Definition 2.18 A subcollection  $B$  of a uniformity  $U$  is a base for  $U$  if and only if each member of  $U$  contains a member of  $B$ . A subcollection  $S$  of  $U$  is a subbase for  $U$  if and only if the collection of finite intersections of members of  $S$  is a base for  $U$ .

Definition 2.19 Let  $u$  be a subset of  $X \times X$  and let  $x$  be in  $X$ . The symbol  $u[x]$  denotes the subset of  $X$  consisting of all  $y$  such that  $(x, y)$  is in  $u$ .

Definition 2.20 Let  $U$  be a uniformity for a set  $X$ . The uniform topology for  $X$  is the collection of all subsets  $V$  of  $X$  such that for each  $x$  in  $V$  there is a  $u$  in  $U$  so that  $u[x] \subset V$ . The pair  $(X, U)$  is called a uniform topological space.

Theorem 2.20 If  $U$  is a uniformity for a set  $X$  and  $A$  is a subset of  $X$  then

$$V = \left\{ v : v = u(A \times A) \text{ for some } u \text{ in } U \right\}$$

is a uniformity for  $A$ .

Proof Clearly the diagonal of  $A \times A$  is contained in each  $v$  in  $V$ . If  $v$  is in  $V$  then  $v = u(A \times A)$  for some  $u$  in  $U$ , and since  $u$  is in  $U$  it follows that  $v$  is in  $V$ . Let  $v_1$  and  $v_2$  be in  $V$ . It follows that  $v_1 v_2 = (u_1 u_2)(A \times A)$  for some members  $u_1$  and  $u_2$  of  $U$ . However,  $u_1 u_2$  is in  $U$  and hence  $v_1 v_2$  is a member of  $V$ . Let  $v = u(A \times A)$  be a member of  $V$  where  $u$  is in  $U$ . Since  $U$  is a uniformity there is an element  $w$  in  $U$  with the property that  $w \cdot w \subset U$ . A straight forward argument by double inclusion yields the following set identity:

$$(w(A \times A)) \cdot (w(A \times A)) = (w \cdot w)(A \times A).$$

But  $w(A \times A)$  is in  $V$  and  $(w \cdot w)(A \times A) \subset u(A \times A) = v$ . Now if  $v$  is in  $V$  and  $v \subset v_1 \subset A \times A$  then  $u(A \times A) \subset v_1 \subset A \times A$  and hence  $u \subset v_1 \cup (X \times X - A \times A)$ . Since  $U$  is a uniformity it follows that  $v_1 \cup (X \times X - A \times A)$  is a member of  $U$  and hence  $v_1$ , its intersection with  $A \times A$ , is in  $V$ . ■

Lemma Let  $X$  be a uniform topological space with uniformity  $U$  and

let  $x$  be in  $X$ . The collection of sets in the form  $u[x]$  for  $u$  in  $U$  is a base for the neighborhood system of  $X$ .

For a sketch of the proof see [3] p. 179.

Theorem 2.21 If  $X$  is a uniform topological space with a uniformity  $U$ , then each subspace  $A$  of  $X$  with the relative topology is a uniform space with the uniformity  $V$  of Theorem 2.20.

Proof By Theorem 2.20 the set

$$V = \left\{ u(A \times A) : u \text{ is in } U \right\}$$

is a uniformity for the set  $A$ . Let  $v$  be in  $V$ , then  $v$  is in the form  $u(A \times A)$  for some  $u$  in  $U$ . A straightforward double inclusion argument yields the following set identity for each  $x$  in  $X$ :

$$v[x] = (u[x])A.$$

Since each  $u[x]$  is a neighborhood of  $x$  in  $X$ , it follows that each  $v[x]$  is a neighborhood of  $x$  in the relative topology of  $A$ . Let  $G$  be an open set in the relative topology of  $A$ . There is an open set  $w$  in  $X$  so that  $G = Aw$ . Since  $w$  is open in the uniform topology for  $X$ , there is a  $u$  in  $U$  so that  $u[x] \subset w$ . Thus  $v[x] = (u[x])A \subset Aw = G$  and therefore  $G$  is open in the uniform topology of  $A$  with uniformity  $V$ . ■

Definition 2.21 A function  $f$  on a uniform topological space  $(X, U)$  into a uniform topological space  $(Y, V)$  is uniformly continuous relative to  $U$  and  $V$  if and only if for each  $v$  in  $V$  the set  $\left\{ (x, y) : (f(x), f(y)) \text{ is in } v \right\}$  is a member of  $U$ .

Theorem 2.22 If  $f$  is a uniformly continuous function on the uniform topological space  $(X, U)$  into the uniform topological space  $(Y, V)$  then for each subset  $A$  of  $X$ , the restriction  $f/A$  is uniformly continuous on  $(A, U^*)$  onto  $(f[A], V^*)$ , where  $U^*$  and  $V^*$  are the relative uniformities of Theorem 2.20.

Proof Let  $v$  be in  $V^*$ . There is some  $w$  in  $V$  so that  $v = w(f[A] \times f[A])$ . Define the set  $u$  in  $U$  by

$$u = \left\{ (x, y) : (f(x), f(y)) \text{ is in } w \right\}.$$

Consider the set  $u^*$  where  $u^*$  is defined by

$$\left\{ (x, y) : (x, y) \in A \times A, \text{ and } ((f/A)(x), (f/A)(y)) \in w(f[A] \times f[A]) \right\}.$$

A short calculation shows that  $u^* = (A \times A)u$  and therefore it follows that  $u^* \in U^*$ . ■

Uniformities for a set  $X$  can be generated by a metric or a pseudo-metric as is outlined in the next definition.

Definition 2.22 Let  $d$  be a pseudo-metric on a set  $X$ . For each positive integer  $r$  let  $V_{d,r}$  denote the collection of all  $(x, y)$  in  $X \times X$  such that  $d(x, y) < r$ . The collection of all  $V_{d,r}$  for a fixed  $d$  is a base for a uniformity for  $X$ , called the uniformity generated by  $d$ . The details verifying that the above collection forms a base are found in [3] p. 184.

Definition 2.23 A uniform space  $(X, U)$  is pseudo-metrizable (metrizable) if and only if  $U$  is the uniformity generated by some pseudo-metric (metric).

Theorem 2.23 If  $(X, U)$  is pseudo-metrizable then each subspace  $A$  of  $X$  is pseudo-metrizable.

Proof Let  $d$  be a pseudo-metric for  $X$  which generates the uniformity  $U$ . Let  $d^*$  be the restriction  $d/(A \times A)$ . Let  $U^*$  be the relative uniformity for  $A$  in Theorem 2.20. Now clearly  $V_{d^*, r} = (V_{d, r})(A \times A)$  for each positive  $r$ . Since each  $V_{d, r}$  is in  $U$  it follows that each  $V_{d^*, r}$  is in  $U^*$ . Let  $u$  be in  $U^*$ . There is some  $w$  in  $U$  so that  $u = w(A \times A)$ . There is a  $V_{d, r}$  with the property that  $V_{d, r} \subset w$  and hence  $V_{d^*, r} = (V_{d, r})(A \times A) \subset w(A \times A) = u$ . Thus the collection of all  $V_{d^*, r}$  is a base for  $U^*$ . ■

The notion of pseudo-metrics generating uniformities admits a generalization to families of pseudo-metrics.

Definition 2.24 Let  $P$  be a family of pseudo-metrics on a set  $X$ . The collection of all  $V_{p, r}$  for  $r > 0$  and  $p$  in  $P$  forms a subbase for a uniformity for  $X$ , called the uniformity generated by  $P$ .

Theorem 2.24 Let  $(X, U)$  be a uniform topological space generated by a family of pseudo-metrics  $P$  and let  $A \subset X$ . The uniform space  $(A, U^*)$  is generated by the family which consists of the members of  $P$  restricted to  $A \times A$ .

A proof of the above theorem can be constructed along the lines of the proof of Theorem 2.23.

Definition 2.25 Let  $A$  be a subset of a uniform topological space  $(X, U)$ . A cover  $C$  of  $A$  is a uniform cover of  $A$  if and only if there is an element  $u$  in  $U$  so that  $u[x]$  is contained in some member of  $C$  for each  $x$  in  $A$ .

Theorem 2.25 Let  $C$  be a uniform cover of the subset  $A$  of  $(X, U)$ . Let  $Y$  be any subset of  $X$  which contains at least one point of  $A$ . The

cover  $C^* = \{cY : c \in C\}$  is a uniform cover in  $(Y, U^*)$  of  $AX$ , where  $U^*$  is the relative uniformity of Theorem 2.20.

Proof There is some  $u$  in  $U$  so that  $u[x] \subset c$  for some  $c$  in  $C$  and for each  $x$  in  $A$ . Let  $v$  be the member of  $U^*$  defined by  $v = u(Y \times Y)$ . It follows that  $v[x] = u[x]Y \subset cY \in C^*$ . ■

A topology on  $X$  induces the product topology on  $X \times X$ . For a definition of the product topology see Definition 5.3.

Definition 2.26 A cover  $C$  of a topological space  $X$  is an even cover if and only if there is a neighborhood  $V$  of the diagonal in  $X \times X$  such that for each  $x$  in  $X$  the set  $V[x]$  is a subset of some member of  $C$ .

Theorem 2.26 If every open cover of  $X$  is even then each closed subset  $A$  of  $X$  inherits this property.

Proof Let  $\{C_a\}$  be an open cover of  $A$  relative to  $A$ . For each  $C_a$  there is an open set  $V_a$  in  $X$  so that  $C_a = AV_a$ . Hence the collection  $F = \{V_a\} \cup \{X - A\}$  is an open cover of  $X$ . Since each open cover of  $X$  is even, there is a neighborhood  $w$  of the diagonal so that for each  $x$  in  $X$  there is some  $V$  in  $F$  so that  $w[x] \subset V$ . But for  $x$  in  $A$  it follows that  $V = V_a$  for some  $V_a$  in  $\{V_a\}$  and thus  $(w(A \times A))[x] = w[x]A \subset AV_a \in \{C_a\}$ . Thus  $w(A \times A)$  is the desired neighborhood of the diagonal in  $A \times A$ . ■

It was remarked earlier that one of the uses for the uniform topology is to study Cauchy sequences. The Cauchy net is a generalization of the Cauchy sequence. In the following two theorems the word "net" may be replaced with the word "sequence."

Definition 2.27 The pair  $(D, \geq)$  is called a directed system if and only if  $D$  is a set and  $\geq$  is a binary relation on  $D$  such that

(a)  $\geq$  is transitive;

(b)  $\geq$  is reflexive;

(c) for each  $m$  and  $n$  in  $D$  there is a  $p$  in  $D$  so that  $p \geq m$  and  $p \geq n$ .

The set  $D$  is called a directed set

Definition 2.28 A net is a function  $S$  whose domain is a directed set.

Definition 2.29 A net  $S$  with domain  $D$  converges to a point  $y$  in the topological space  $X$  if and only if the range of  $S$  is contained in  $X$  and for each open set  $U$  containing  $y$  there is an element  $m$  in  $D$  such that  $S(n)$  is in  $U$  for each  $n \geq m$ .

Theorem 2.27 Let  $A$  be a subset of the topological space  $X$ . If  $S$  is a net in  $A$  and if  $S$  converges to the point  $s$  in  $A$  relative to the topology of  $X$  then  $S$  converges to  $s$  relative to the subspace topology of  $A$ .

The proof is left for the reader.

Definition 2.30 Let  $S$  be a net with directed set  $D$  and whose range is a subset of the uniform topological space  $(X, U)$ . The net  $S$  is a Cauchy net if and only if for each  $u$  in  $U$  there is an element  $p$  in  $D$  such that  $(S(n), S(m))$  is in  $u$  for each pair  $m, n$  such that  $m \geq p$  and  $n \geq p$ .

Theorem 2.28 Let  $A$  be a subset of a uniform topological space  $(X, U)$ . If  $S$  is a net whose range is a subset of  $A$  and if  $S$  is a Cauchy net relative to  $U$  then  $S$  is a Cauchy net relative to  $U^*$ , the relative uniformity of  $A$  given in Theorem 2.20.

Proof Let  $v$  be in  $U^*$ . There is some  $u$  in  $U$  so that  $v = u(A \times A)$ . There is a  $p$  in  $D$  so that for each  $m$  and  $n$  such that  $m \geq p$  and  $n \geq p$  then  $(S(n), S(m))$  is in  $u$ . However, each  $(S(n), S(m)) \subset A \times A$  and hence  $(S(n), S(m))$  is in  $v$  for  $m \geq p$  and  $n \geq p$ .  $\square$



Definition 2.31 A uniform topological space  $X$  is called complete if and only if each Cauchy net in  $X$  converges to a point of  $X$ .

Theorem 2.29 Each closed subspace of a complete uniform topological space is complete.

Proof The proof follows from the fact that convergence points of a net are in the closure of its range. ■

Definition 2.32 A uniform topological space  $(X, U)$  is totally bounded if and only if for each  $u$  in  $U$  the set  $X$  is the union of a finite number of sets  $B$  with the property that  $B \times B \subset u$ .

Theorem 2.30 If  $(X, U)$  is totally bounded then each subspace  $A$  of  $X$  with the relative uniformity of Theorem 2.20 is totally bounded.

Proof Let  $(A, U^*)$  be any subspace of  $(X, U)$ . Let  $u = v(A \times A)$  for some  $v$  in  $U$  be a typical element of  $U^*$ . Since  $X$  is totally bounded there is a finite number of sets  $B_1, \dots, B_n$  such that  $B_k \times B_k \subset v$  for  $k=1, 2, \dots, n$ , and  $X = \bigcup_{k=1}^n B_k$ . Let  $B_k^* = B_k \cap A$  for  $k=1, \dots, n$ . It follows that  $B_k^* \times B_k^* = B_k \times B_k \cap (A \times A) \subset u$  and  $\bigcup_{k=1}^n B_k^* = A$ . ■

## CHAPTER III

### FUNCTIONS

In this chapter many varieties of functions on topological spaces are studied in order to find out whether or not a function possessing a specific property maintains that property when restricted to a subset of the space. If such a function does not maintain that property, an attempt is made to determine what additional restrictions on the subsets are necessary in order to do so.

Definition 3.1 A function  $f$  on a topological space  $X$  into a topological space  $Y$  is continuous if and only if for each open set  $V$  in  $Y$ , the inverse set  $f^{-1}[V]$  is open in  $X$ .

Theorem 3.1 Let  $f$  be a continuous function on a topological  $X$  into a topological space  $Y$  and let  $A$  be any subset of  $X$ . The restriction  $f/A$  is continuous on  $A$  into  $Y$ .

Proof Let  $V$  be open in  $Y$ . Then  $(f/A)^{-1}[V] = Af^{-1}[V]$  which is an open set in the topology of  $A$ . ■

Corollary If  $f$  is a homeomorphism on  $X$  onto  $Y$  then  $f/A$  is a homeomorphism on  $A$  onto  $f[A]$  for each subset  $A$  of  $X$ .

Definition 3.2 A function  $f$  on the topological space  $X$  onto the topological space  $Y$  is open(closed) if and only if  $f$  maps each open(closed) set in  $X$  onto open(closed) sets in  $Y$ .

Theorem 3.2 Let  $f$  be an open(closed) function on a topological space  $X$  into a topological space  $Y$ . For each open(closed) subset  $A$  of  $X$ ,

$f|_A$  is open(closed).

Proof Suppose  $f$  is open. Let  $u$  be open in  $A$ . Since  $A$  is open in  $X$  it follows that  $u$  is open in  $X$  and thus  $f[u]$  is open in  $Y$ . A similar argument holds for  $f$  closed. ■

Theorem 3.3 If  $f$  is a one to one open(closed) function on a topological space  $X$  into a topological space  $Y$  then the restriction  $f|_A$  is open (closed) for each subspace  $A$  of  $X$ .

Proof The theorem follows from the fact that for one to one functions  $f$ ,  $f[A \cap B] = f[A] \cap f[B]$ . ■

Lemma Let  $f$  be a function on a set  $X$  into a set  $Y$ . If  $A$  is an inverse set then  $f[AB] = f[A]f[B]$  for all subsets  $B$  of  $X$ .

The proof follows quickly using a double inclusion argument.

Theorem 3.4 Let  $f$  be a function on a topological space  $X$  into a topological space  $Y$  and let  $A$  be an inverse set. The restriction  $f|_A$  is open(closed) if  $f$  is open(closed).

The proof follows from an application of the preceding lemma.

Definition 3.3 A continuous function  $f$  from a topological space  $X$  onto a topological space  $Y$  is compact if and only if the inverse of every compact set is compact.

Theorem 3.5 Let  $f$  be a compact function on the Hausdorff topological space  $X$  onto the topological space  $Y$ . For each closed subset  $A$  of  $X$ ,  $f|_A$  is compact.

Proof Let  $K$  be a compact subset of  $f[A]$ . Since  $f^{-1}[K]$  is compact and  $X$  is Hausdorff it follows that  $f^{-1}[K]$  is closed. Thus  $(f|_A)^{-1}[K] = Af^{-1}[K]$ , and is therefore compact. ■

Theorem 3.6 Let  $f$  be a compact function on the topological space  $X$

onto the topological space  $Y$  and let  $A$  be an inverse set, then  $f/A$  is a compact function.

Proof Let  $K$  be a compact subset of  $f[A]$ . By Theorem 2.14  $K$  is compact in  $Y$  and hence  $f^{-1}[K]$  is compact in  $X$ . But since  $A$  is an inverse set,  $(f/A)^{-1}[K] = f^{-1}[K]$  and therefore by Theorem 2.14 it follows that  $(f/A)^{-1}[K]$  is compact in  $A$ . ■

Definition 3.4 Let  $f$  be a function on a topological space  $X$  onto a topological space  $Y$ . The function  $f$  is quasi-compact if and only if for each open set in the form  $f^{-1}[V]$ , the set  $V$  is open in  $Y$ .

Theorem 3.7 If  $f$  is a quasi-compact function on the topological space  $X$  onto the topological space  $Y$  and  $A$  is an open inverse set then  $f/A$  is quasi-compact.

Proof There is a subset  $T$  of  $Y$  such that  $A = f^{-1}[T]$ . Let  $B = (f/A)^{-1}[V]$  be open in  $A$  where  $V \subset T$ . Then  $B = f^{-1}[V]$  and  $B$  is open in  $X$  since  $A$  is open in  $X$ . Now  $f$  is quasi-compact. Therefore  $V$  is open in  $Y$  and, consequently, is open in  $T$ . ■

The following counterexamples illustrate that the conditions on  $A$  in the previous theorem are not superfluous.

Example 3.1 Let  $X = \{1, 2, 3, 4\}$  and let the open sets be  $\{1\}$ ,  $\{2\}$ ,  $\{1, 2\}$ ,  $X$ , and  $\emptyset$ . Let  $Y = \{I, II\}$  with the indiscrete topology. Define  $f$  by  $f(1) = f(3) = I$  and  $f(2) = f(4) = II$ . The function  $f$  is quasi-compact. Let  $A$  be the open set  $\{1, 2\}$ . Now  $f^{-1}[I] = 1$  is an open inverse set, but  $f(1)$  is not open in  $f[A]$ . The trouble here is that  $A$  is not an inverse set.

Example 3.2 Let  $X = \{1, 2, 3, 4\}$  and let the open sets by  $\{1\}$ ,  $\{1, 2\}$ ,  $X$  and  $\emptyset$ . Let  $Y = \{I, II, III\}$  have the indiscrete topology. Define  $f$  by

$f(1) = f(4) = I$ ,  $f(2) = II$ , and  $f(3) = III$ . Let  $A = f^{-1}[\{II, III\}] = \{2, 3\}$ . Now  $\{2\} = f^{-1}[\{II\}]$  is open in  $A$  but  $f(2) = \{II\}$  is not open in  $Y$ . The trouble here is that  $A$  is not open in  $Y$ .

Definition 3.5 Let  $f$  be a function on a topological space  $X$  onto a topological space  $Y$ . The function  $f$  is quasi-open if and only if for each  $y$  in  $Y$  and for each compact component  $K$  of  $f^{-1}(y)$  and for each open set  $U$  containing  $K$ ,  $f[U]$  is a neighborhood of  $y$  in  $Y$ .

Theorem 3.8 Let  $f$  be a quasi-open function on the topological space  $X$  into the topological space  $Y$ . If  $A$  is an inverse set then  $f/A$  is a quasi-open function on  $A$  onto  $f[A]$ .

Proof Let  $y$  be in  $f[A]$  and let  $K$  be a compact component of  $f^{-1}(y)$  in  $A$ . By Theorems 2.3 and 2.4 it follows that  $K$  is also a compact component of  $f^{-1}(y)$  in  $X$ . Let  $U$  be an open set in  $A$  containing  $K$ . There is an open set  $V$  in  $X$  such that  $U = AV$ . Now  $f(V)$  is a neighborhood of  $y$  in  $Y$  and  $f(VA) = f(V)f(A)$  since  $A$  is an inverse set. Hence  $f(U)$  is a neighborhood of  $y$  in  $f[A]$ . ■

Definition 3.6 A function  $f$  on a topological space  $X$  into a topological space  $Y$  is monotone if and only if for each  $y$  in  $Y$ ,  $f^{-1}(y)$  is connected.

Theorem 3.9 Let  $f$  be a monotone function on the topological space  $X$  into the topological space  $Y$  and let  $A$  be any inverse set, then  $f/A$  is monotone.

Proof Let  $y$  be in  $f[A]$ , then  $f^{-1}(y) = (f/A)^{-1}[y]$ . Now  $f^{-1}(y)$  is connected in  $X$  and hence by Theorem 2.4  $f^{-1}(y)$  is connected in  $A$ . ■

Definition 3.7 Let  $f$  be a continuous function on the topological space  $X$  onto the topological space  $Y$ . The function  $f$  is quasi-monotone

if and only if for each continuum  $K$  with non-empty interior in  $Y$ ,  $f^{-1}[K]$  has a finite number of components, each mapping onto  $K$  under  $f$ .

Lemma Let  $f$  be a continuous function on the locally connected continuum  $X$  onto  $Y$ , then  $f$  is quasi-monotone if and only if each component of the inverse of each region  $R$  (i.e. open, connected) in  $Y$  maps onto  $R$  under  $f$ .

For a proof see [6], p. 152.

Theorem 3.10 Let  $X$  be a locally connected continuum and let  $f: X \rightarrow Y$  be a quasi-monotone map onto  $Y$ , and let  $B$  be a region in  $Y$ , then if  $A$  is any component of  $f^{-1}[B]$ ,  $f/A$  is quasi-monotone.

Proof By the above lemma  $f[A] = B$ . Let  $K$  be any continuum of  $f[A]$ , with non-empty interior relative to  $f[A]$ . Then  $K$  is a continuum with a non-empty interior in  $Y$  since  $B$  is open in  $Y$ . Hence  $(f/A)^{-1}[K] = f^{-1}[K]A$ . The components of  $(f/A)^{-1}[K]$  will be those of  $f^{-1}[K]$  which are contained in  $A$  since  $A$  is a component and thus there will be only finitely many, each of which map onto  $K$  under  $f/A$ . ■

Theorem 3.11 Let  $f$  be a quasi-monotone function on the topological space  $X$  onto the topological space  $Y$  and let  $B$  be an open set in  $Y$ . If  $A = f^{-1}[B]$  then  $f/A$  is a quasi-monotone function.

Proof Let  $K$  be any continuum of  $B$  with non-empty interior relative to  $B$ . By the corollary to Theorem 2.3 and the Theorem 2.14,  $K$  is also a continuum in  $Y$  and furthermore  $K$  has a non-empty interior relative to  $Y$  since  $B$  is open in  $Y$ . Hence  $(f/A)^{-1}[K]$  has only a finite number of components, each of which map onto  $K$  under  $f/A$ . ■

Definition 3.8 A function  $f$  on a topological space  $X$  into a topological space  $Y$  is called light if and only if the inverse of each point in  $Y$  is totally disconnected.

Theorem 3.12 If  $f$  is a light function on a topological space  $X$  into a topological space  $Y$  then  $f/A$  is a light function for each subset  $A$  of  $X$ .

Proof Let  $y$  be in  $Y$ . The set  $(f/A)^{-1}[y]$  is a subset of  $f^{-1}[y]$  which is totally disconnected in  $Y$  and thus by the corollary to Theorem 2.3,  $(f/A)^{-1}[y]$  is totally disconnected. ■

Definition 3.9 Let  $X$  be a topological space and let  $Y$  be a subset of  $X$ . The function  $f$  on  $X$  onto  $Y$  is a retraction of  $X$  onto  $Y$  if and only if  $f$  is continuous and for each  $y$  in  $Y$   $f(y) = y$ .

Theorem 3.13 Let  $f$  be a retraction of the topological space  $X$  onto  $Y$ . If  $A$  is any subset of  $X$  containing  $Y$  then  $f/A$  is a retraction of  $A$  onto  $Y$ .

The proof is simple and is left for the reader.

Theorem 3.14 Let  $f$  be a retraction of the topological space  $X$  onto  $Y$ . Let  $A$  be any inverse set, then  $AY \neq \emptyset$  and  $f/A$  is a retraction of  $A$  onto  $AY$ .

Proof The continuity of  $f/A$  follows from Theorem 3.1. By the lemma preceding Theorem 3.4 it follows that  $f[AY] = f[A]f[Y]$  and hence  $f[A]Y = f[A] = f[AY] = AY$ . ■

## CHAPTER IV

## DECOMPOSITION SPACES

The properties of a topological space  $X$  which are inherited by various decomposition spaces of  $X$  and the relative decomposition for subspaces are the main subjects in this chapter with emphasis placed on upper semi-continuous decompositions.

Definition 4.1 Let  $f$  be a function on a topological space  $X$  onto a set  $Y$ . The quotient topology for  $Y$  relative to  $f$  is the topology consisting of all the subsets of  $Y$  which have open inverses under  $f$ .

For the details in verifying that the quotient topology is indeed a topology see [3], p. 94.

Definition 4.2 Let  $X$  be a set. A collection  $D$  of pair-wise disjoint subsets of  $X$  is a decomposition of  $X$  if and only if the union of the members of  $D$  is  $X$ .

Definition 4.3 Let  $X$  be a set with a decomposition  $D$ . The projection map  $P$  is the function defined on  $X$  with range  $D$  whose value at each  $x$  in  $X$  is the member of  $D$  containing  $x$ .

Definition 4.4 Let  $X$  be a topological space and let  $D$  be a decomposition of  $X$ . The decomposition space  $X/D$  is the set  $D$  with the quotient topology relative to the projection map  $P$ .

Definition 4.5 Let  $X$  be a topological space. If a decomposition  $D$  consists of the equivalence classes of a relation  $R$  on  $X \times X$ , then the decomposition space  $X/D$  is written  $X/R$ .



The following, through Theorem 4.7, is a summary of the paper in [4].

Let  $X$  and  $X^*$  be topological spaces with equivalence relations  $R$  and  $R^*$  respectively. Let  $f: X \rightarrow X^*$  be a function such that if  $(x, y)$  is in  $R$  then  $(f(x), f(y))$  is in  $R^*$ . A function  $f^*: X/R \rightarrow X^*/R^*$  is induced as follows:

$$f^*[R[x]] = R^*[f(x)] \quad \text{for each } R[x] \text{ in } X/R,$$

where  $R[x]$  denotes the equivalence class containing  $x$ .

It is natural to ask "what properties of  $f$  are inherited by  $f^*$ ?"

Theorem 4.1 If  $f$  is continuous then  $f^*$  is continuous.

Definition 4.6 The function  $f$  is 1-1( $R, R^*$ ) if and only if  $(x, y)$  is not in  $R$  implies  $(f(x), f(y))$  is not in  $R^*$ .

Theorem 4.2 If  $f$  is 1-1( $R, R^*$ ) and onto then  $f$  is open(closed) implies  $f^*$  is open(closed).

Theorem 4.3 If  $f$  is a homeomorphism then  $f^*$  is homeomorphism if and only if  $f$  is 1-1( $R, R^*$ ).

Theorem 4.4 If  $X^*$  has the quotient topology then  $X^*/R^*$  has the quotient topology relative to  $f^*$ .

Theorem 4.5 Let  $X$  be compact and let  $X/R$  be Hausdorff. If  $f$  preserves compactness then  $f^*$  preserves compactness.

Theorem 4.6 Let  $f$  be 1-1( $R, R^*$ ) and onto. If the projection map  $P$  is closed, and if  $f^{-1}[A^*]$  and  $f^{-1}[B^*]$  are separated whenever  $A^*$  and  $B^*$  are separated in  $X^*$  then  $f^*$  preserves connectedness.

Theorem 4.7 Let  $f$  be a 1-1( $R, R^*$ ) and onto function. If  $P^*$  is closed and if  $f$  preserves the property of being separated then so does  $f^*$ .

The proofs of the preceding seven theorems are in [4].

The quotient topology (Definition 4.1) on  $X/D$  forces the projection map  $P$  to be continuous which yields the next theorem, considering  $X/D$  as a continuous image of  $X$ .

Theorem 4.8 If the topological space  $X$  with decomposition  $D$  is compact, connected, Lindelöf, or separable then  $X/D$  is compact, connected, Lindelöf, or separable, respectively.

The following theorem appears as an exercise in [3].

Theorem 4.9 The decomposition space  $X/D$  is  $T_1$  if and only if the members of  $D$  are closed in  $X$ .

Proof Suppose  $X/D$  is  $T_1$ . Each  $P[K]$  is closed for each  $K$  in  $D$  since points of  $T_1$  spaces are closed. Now  $P^{-1}P[K] = K$  and hence  $K$ , the inverse of a closed set under a continuous map, is closed.

Suppose each  $K$  in  $D$  is closed in  $X$ . Let  $H$  and  $K$  be distinct members of  $D$ . The complement  $X - H$  is an open set containing  $K$ . Now  $P[X - H]$  is an open set in the quotient topology of  $X/D$  since  $P^{-1}P[X - H] = X - H$ . Furthermore,  $P[X - H]$  contains  $K$  and does not intersect  $H$ . Similarly,  $P[X - K]$  is a neighborhood of  $P[H]$  not containing  $P[K]$ . ■

Theorem 4.10 Let the set  $X$  have a decomposition  $D$  and let  $A$  be a subset of  $X$ . The collection  $D^* = \{KA : K \in D\}$  is a decomposition for  $A$ .

The proof is a simple exercise in set algebra.

Definition 4.7 The set  $D^*$  in Theorem 4.10 is called the relative decomposition for  $A$ . If  $X$  is a topological space then  $A/D^*$  is called the subdecomposition space for  $A$ .

Definition 4.8 A decomposition  $D$  of a topological space  $X$  is upper semi-continuous (u.s.c.) if and only if for each open set  $U$  in  $X$  the union

of the sets in  $D$  contained in  $U$  is open in  $X$ . Equivalently,  $D$  is u.s.c. if and only if for each closed set  $F$  in  $X$  the union of sets in  $D$  intersecting  $F$  is closed. For the definition of a lower semi-continuous (l.s.c.) decomposition interchange the words open and closed.

Theorem 4.11 Let  $D$  be an u.s.c. or a l.s.c. decomposition for the topological space  $X$  and let  $A$  be the union of the members of some subcollection  $C$  of  $D$ . If  $D^*$  is the relative decomposition for  $A$  then  $A/D^*$  is identically  $C$  with the relative topology in  $X/D$ .

Proof The set  $D^* = \{KA: K \in D\} = \{K: K \in C\} = C$ . Let  $U$  be open in  $C$  with the relative topology in  $X/D$ . There is an open set  $V$  in  $X/D$  such that  $U = VC$ . By the definition of quotient topology  $P^{-1}[V]$  is open in  $X$ . Hence  $P^{*-1}[V] = P^{-1}[V]A$  which is open in  $A$  with the relative topology of  $A$  in  $X$ , and thus  $U$  is open in  $A/D^*$ .

Suppose  $D$  is u.s.c. Let  $U$  be open in  $A/D^*$ . It is easy to show that  $P^{*-1}[U] = P^{-1}[U]A = P^{-1}[U]$  from the structure of  $A$ . Now  $P^{-1}[U]$  is open in  $A$  with the relative topology in  $X$  since  $P^{*-1}[U]$  is open in  $A$ , and therefore  $P^{-1}[U] = WA$  for some open set  $W$  in  $X$ . Let  $G = \left\{K: K \in D \text{ and } K \subset W\right\}$ . The set  $P^{-1}[G]$  is open in  $X$  since  $D$  is u.s.c. and moreover  $GC = U$ . Thus  $U$  is open in  $C$  with the relative topology in  $X/D$ .

If  $D$  is l.s.c. then an analogous argument based on the closed sets of each topology gives the conclusion. ■

Theorem 4.12 Let  $D$  be an u.s.c. decomposition of the topological space  $X$  and let  $A$  be a closed subset of  $X$ . The relative decomposition  $D^*$  for  $A$  is an u.s.c. decomposition.

Proof Let  $F$  be a closed subset of  $A$ . The set  $F$  is closed in  $X$  and hence the union  $V = \bigcup \left\{d: d \in D \text{ and } dF \neq \emptyset\right\}$  is closed in  $X$ . Let  $V^* =$

$\bigcup \{d^*: d^* \in D^* \text{ and } d^* \cap F \neq \emptyset\}$ . A simple calculation yields  $V^* = AV$  and therefore  $V^*$  is closed in  $A$ . ■

Replacing closed sets by open sets in Theorem 4.12 yields the dual theorem:

Theorem 4.13 If  $D$  is a l.s.c. decomposition of the topological space  $X$  and if  $A$  is an open subset of  $X$  then the relative decomposition  $D^*$  for  $A$  is a l.s.c. decomposition.

Theorem 4.14 Let  $D$  be an u.s.c.(l.s.c.) decomposition of a topological space  $X$ . If  $A$  is an inverse set under the projection map  $P$  then the relative decomposition  $D^*$  for  $A$  is an u.s.c.(l.s.c.) decomposition.

Proof Suppose  $D$  is an u.s.c. decomposition. Let  $U$  be an open set in  $A$  and let  $V = \bigcup \{d^*: d^* \in D^* \text{ and } d^* \subset U\}$ . It follows that  $V = \bigcup \{dA: d \in D \text{ and } dA \subset U\}$ . However, either  $dA = d$  or  $dA = \emptyset$  for each  $d$  in  $D$  since  $A$  is an inverse set. Thus  $V = \bigcup \{d: d \in D \text{ and } d \subset U\}$  and therefore  $V$  is open in  $X$  by the upper semi-continuity of  $D$  which proves that  $V$  is open in  $A$ .

If  $D$  is l.s.c. the proof is similar. ■

The following theorem is a modification of Theorem 3-33 in [1].

Theorem 4.15 If  $X$  is a compact Hausdorff space and if  $D$  is an u.s.c. decomposition of  $X$  such that each element of  $D$  is a closed subset of  $X$  then  $X/D$  is a compact Hausdorff space.

Proof The compactness follows from Theorem 4.8. Each compact Hausdorff space is normal (see [3], p. 141). Let  $H$  and  $K$  be a disjoint pair of members of  $D$ . There are open disjoint neighborhoods  $U$  and  $V$  of  $H$  and  $K$  respectively. Let  $U^* = \bigcup \{d: d \in D \text{ and } d \subset U\}$  and let  $V^* = \bigcup \{d: d \in D \text{ and } d \subset V\}$ . Since  $D$  is u.s.c. then  $U^*$  and  $V^*$  are open disjoint

neighborhoods of  $H$  and  $K$  respectively. Hence  $P[U^*]$  and  $P[V^*]$  are open disjoint neighborhoods of  $H$  and  $K$ , respectively, in  $X/D$ . ■

Lemma If  $X$  is a regular topological space and if  $U$  is a neighborhood of the compact subset  $A$  of  $X$  then there is a neighborhood  $V$  of  $A$  such that  $A \subset V \subset \bar{V} \subset U$ , where  $\bar{V}$  denotes the closure of  $V$ . See [3], p. 141 for a proof.

Theorem 4.16 If  $X$  is a regular topological space and if  $D$  is an u.s.c. decomposition of  $X$  whose elements are compact subsets of  $X$  then  $X/D$  is a regular topological space.

Proof Let  $d$  be an element of  $D$  and let  $U$  be any open set in  $X/D$  which contains  $d$ . The set  $P^{-1}[d]$  is compact in  $X$  and  $P^{-1}[U]$  is a neighborhood in  $X$  of  $P^{-1}[d]$ . Thus by the lemma there is an open set  $V$  in  $X$  so that  $P^{-1}[d] \subset V \subset \bar{V}$  and  $\bar{V} \subset P^{-1}[U]$ . Let  $W = \bigcup \left\{ K : K \in D \text{ and } K \subset V \right\}$ . The set  $W$  is open in  $X$  and since  $W \subset V$  and  $\bar{W} \subset \bar{V}$  then  $P^{-1}[d] \subset W \subset \bar{W} \subset \bar{V} \subset P^{-1}[U]$ . Now  $P[W]$  is a neighborhood of  $d$  since  $W$  is an open inverse set. Let  $c$  be any point in  $X/D - U$ . It follows that  $P^{-1}[c] \subset X - P^{-1}[U] \subset X - \bar{W}$ . Let  $T$  be the open set  $X - \bar{W}$  and let  $S = \bigcup \left\{ d : d \in D \text{ and } d \subset T \right\}$ . The set  $S$  is an open inverse set. Hence  $P[S]$  is an open set in  $X/D$  and therefore  $c \in P[S] \subset P[X - \bar{W}] \subset P[X - W] = X/D - P[W]$ . The set  $P[S]$  is a neighborhood of  $c$  in  $X/D$  not intersecting  $P[W]$  and thus  $\overline{P[W]} \subset U$ . ■

Theorem 4.17 If  $X$  is a topological space satisfying the second axiom of countability and if  $D$  is an u.s.c. decomposition of  $X$  whose elements are compact subsets of  $X$  then  $X/D$  satisfies the second axiom of countability.

Proof Let  $\left\{ B_n \right\}$  be a countable base for  $X$ . Let  $C$  be the countable

collection consisting of all the sets in the form  $\bigcup_{i=1}^n B_{a_i}$  for each finite set of positive integers  $\{a_1, \dots, a_n\}$ . For each  $c$  in  $C$  define  $c^*$  by  $c^* = \bigcup \{d: d \in D \text{ and } d \subset c\}$ . Let  $C^*$  be the collection of all the  $c^*$ 's. Define the countable collection  $F$  of open sets in  $X/D$  by  $F = \{P[c^*] : c^* \in C^*\}$ . Let  $U$  be an open set in  $X/D$  and let  $d$  be in  $U$ . For each  $x$  in  $d$  there is a  $B_{n_x}$  in  $\{B_n\}$  so that  $x \in B_{n_x} \subset P^{-1}[U]$ . The  $B_{n_x}$ 's form an open cover of the compact set  $d$  and hence there is a finite subcover  $\{B_{m_1}, \dots, B_{m_k}\}$  of  $d$ . Let  $c$  be the element of  $C$  which is the union of the  $B_{m_i}$ 's. For the corresponding  $c^*$ ,  $P[c^*]$  is a neighborhood in  $X/D$  of  $d$  contained in  $U$ . ■

Lemma (Urysohn's metrization theorem) Each regular  $T_1$  topological space satisfying the second axiom of countability is metrizable. See [3] for a proof.

Theorem 4.18 Let  $X$  be a topological space with an u.s.c. decomposition  $D$  whose elements are compact subsets of  $X$ . If  $X$  is a separable metric space then so is  $X/D$ .

Proof Each separable metric space is regular, Hausdorff, and satisfies the second axiom of countability. Hence the elements of  $D$  are closed in  $X$ . By Theorems 4.9, 4.16, and 4.17, the space  $X/D$  is  $T_1$ , regular, and satisfies the second axiom of countability. The metrizability of  $X/D$  follows from Urysohn's theorem and separability follows from Theorem 4.8. ■

The preceding three theorems are modifications of the ideas found in G. T. Whyburn's proof of Theorem 4.18 (see [6], p. 123).

It is natural to ask which decompositions are u.s.c. Whyburn gives

the example in [6] of the decomposition of a compact space by its components. The next two theorems provide more examples. They are generalizations of two theorems found in [6] and the proofs are original.

Theorem 4.19 Let  $f$  be a continuous function on the compact topological space  $X$  onto the Hausdorff space  $Y$ . Let  $D$  be the decomposition of  $X$  into the inverse sets of points in  $Y$ , then  $D$  is an u.s.c. decomposition for  $X$ .

Proof Let  $C$  be a closed subset of  $X$  and let  $A$  be the set  $\bigcup \{d: d \in D \text{ and } dC \neq \emptyset\}$ . Suppose that  $A$  is not closed. Then there is a limit point  $p$  of  $A$  which does not belong to  $A$ , and consequently does not belong to  $C$ . There is a net  $S$  with domain  $E$  directed by  $\geq$  and with range contained in  $A$  which converges to  $p$ . Since  $f$  is continuous, the net  $f(S)$  converges to  $f(p)$ . Let  $d_n = P[S(n)]$  for each  $n$  in  $E$  and choose an element  $r_n$  in each  $d_n C$ . Since  $C$  is closed subset of a compact space,  $C$  is compact and therefore there is a subnet  $r_{n_k}$  which converges to some  $r$  in  $C$ . Now the net  $f(r_{n_k})$  converges to  $f(r)$  in  $f[C]$ . Since the  $d_n$ 's are point inverses under  $f$ , it follows that  $f(S(n_k)) = f(r_{n_k}) = f[d_{n_k}]$  and therefore  $f(r_{n_k})$  converges to  $f(p)$ . In a Hausdorff space the convergence of a net is unique and thus  $f(r) = f(p)$ . Hence  $r \in Cf^{-1}f(p)$  so that  $Cf^{-1}(f(p)) \neq \emptyset$  and therefore  $f^{-1}(f(p)) \subset A$  which is a contradiction since  $p$  is not in  $A$ . ■

Corollary If the compact space  $X$  and the space  $Y$  are separable metric spaces,  $f$  is a continuous function on  $X$  onto  $Y$ , and  $D$  is the decomposition of  $X$  into the inverse sets of the points of  $Y$  then  $X/D$  is a separable metric space.

Proof The decomposition  $D$  is u.s.c. by Theorem 4.19. The points

of  $Y$  are closed hence the elements of  $D$  are closed subsets of a compact space and hence compact. Theorem 4.18 yields the conclusion. ■

Theorem 4.20 Let the topological space  $X$  be Hausdorff and locally compact and let  $Y$  be a Hausdorff topological space. If  $f$  is a continuous function on  $X$  onto  $Y$  such that for each  $y$  in  $Y$   $f^{-1}(y)$  is a continuum, then the decomposition  $D$  of  $X$  into the point inverses of  $Y$  is u.s.c.

Proof Let  $C$  be a closed subset of  $X$  and let  $A$  be the set  $\bigcup \{d : d \in D \text{ and } dC \neq \emptyset\}$ . Suppose that  $A$  is not closed. Then there is limit point  $p$  of  $A$  which does not belong to  $A$ . Each point of  $f^{-1}(f(p))$  can be covered with a closed compact neighborhood not intersecting  $C$  and consequently there is a finite subcover  $K$  of  $f^{-1}(f(p))$  since it is compact and such that  $KC = \emptyset$ . There is a net  $S$  with domain  $E$  and range contained in  $A$  which converges to  $p$ . Let  $d_n = P[S(n)]$  for each  $n$  in  $E$ . Since  $K$  is a neighborhood of  $p$  there is an  $N$  in  $E$  so that for each  $n \geq N$   $S(n) \in K$ . Now each  $d_n B(K) \neq \emptyset$  for each  $n \geq N$ , otherwise  $d_n K$  and  $d_n(X - K)$  would be a separation for  $d_n$ , where  $B(K)$  denotes the boundary of  $K$ . Choose an  $r(n)$  in  $d_n B(K)$  for each  $n \geq N$ . The set  $B(K)$  is a closed subset of the compact set  $K$  and is therefore compact. Thus there is a subnet of  $r(n)$ , say  $r(n(k))$ , which converges to a point  $r$  in  $B(K)$ . The continuity of  $f$  implies that the net  $f(r(n(k)))$  converges to  $f(r)$ . But since each  $r(n(k))$  and  $S(n(k))$  are in the same point inverse set  $d_{n(k)}$  it follows that  $f(r(n(k))) = f(S(n(k)))$  and therefore  $f(r(n(k)))$  converges to  $f(p)$  and  $f(p) = f(r)$ . Thus it follows that  $f^{-1}(f(p))B(K)$  contains  $r$  which is a contradiction since  $f^{-1}(f(p))$  lies in the interior of  $K$ . ■

Corollary Let the topological space  $X$  be Hausdorff, and locally compact and let  $Y$  be a Hausdorff topological space. Let  $f$  be a continuous



function on  $X$  into  $Y$  such that for each  $y$  in  $Y$ ,  $f^{-1}(y)$  is a continuum and let  $D$  be the decomposition of  $X$  into the point inverses of  $Y$  under  $f$ . If  $X$  is separable metric then so is  $X/D$ .

Proof The decomposition is u.s.c. by Theorem 4.20. The points of  $D$  are compact in  $X$  since they are continua. Thus, by Theorem 4.18,  $X/D$  is separable metric. ■

Theorem 4.21 If  $D$  is an u.s.c. decomposition of the locally compact topological space  $X$ , then  $X/D$  is locally compact, where the members of  $D$  are compact.

Proof Let  $d$  be a member of  $D$ . For each  $x$  in  $d$  there is a compact neighborhood  $C_x$  of  $x$ . Let  $U_x$  denote the interior of  $C_x$  for each  $x$  in  $d$ . There is a finite subcover  $\{U_{x_1}, \dots, U_{x_n}\}$  of  $d$ . Let  $V = \bigcup \{d : d \in D \text{ and } d \subset \bigcup_{i=1}^n U_{x_i}\}$ . The set  $P[V]$  is an open subset of the compact set  $P[\bigcup_{i=1}^n C_{x_i}] = C^*$ . Hence  $C^*$  is a compact neighborhood of  $d$ . ■

Theorem 4.22 If the locally connected topological space  $X$  has an u.s.c. decomposition  $D$ , then  $X/D$  is locally connected.

Proof Let  $d$  be a member of  $D$  and let  $U$  be an open set in  $X/D$ , containing  $d$ . The set  $P^{-1}[U]$  is an open set in  $X$  which contains  $d$ . For each  $x$  in  $d$  the component  $C_x$  of  $P^{-1}[U]$  to which  $x$  belongs is a neighborhood of  $x$ . Let the set  $W$  be the union  $\bigcup \{C_x : x \in d\}$ . Then  $W$  is a neighborhood of  $d$  contained in  $P^{-1}[U]$ . Let  $M = \bigcup \{d : d \in D \text{ and } d \subset W^0\}$  where  $W^0$  denotes the interior of  $W$  in  $X$ . Now  $d \subset M \subset W \subset P^{-1}[U]$  and  $M$  is open by the u.s.c. of  $D$ . Also  $P[W] = \bigcup \{P[C_x] : x \in d\}$  and each  $P[C_x]$  is connected and contains the common point  $d$ . Hence  $P[W]$  is connected and thus is a subset of the component of  $U$  to which  $d$  belongs. The open set  $P[M]$  is contained in  $W$

and therefore the component to which  $d$  belongs is a neighborhood of  $d$  in  $X/D$ . ■

Theorem 4.23 Let  $X$  and  $Y$  be compact Hausdorff spaces, let  $f$  be a continuous function on  $X$  onto  $Y$ , and let  $D$  be the decomposition of  $X$  into the components of the inverses of points in  $Y$  under  $f$ . Then  $f$  has a  $m$ -1 decomposition; i.e., there is a monotone map  $m$  (the proj. map  $P$ ) and a light map  $L(f(m^{-1}))$  such that  $f = L(m)$ .

For the proof see [1], p. 137.

Corollary Each closed subspace  $A$  of the space  $X$  in Theorem 4.23 allows  $f/A$  to have a  $m$ -1 decomposition.

## CHAPTER V

## PRODUCT SPACES

Given a collection of topological spaces, a topology can be defined on their Cartesian product. The induced properties are the properties of each space in the collection which are inherited by the product space and conversely.

Definition 5.1 Let  $Q$  be the collection of non-empty sets  $\{X_a : a \in A\}$ . The Cartesian product of the members of  $Q$ ,  $\prod \{X_a : a \in A\}$ , is the collection of all functions with domain  $A$  and with range contained in  $\bigcup \{X_a : a \in A\}$  such that for each  $a$  in  $A$ ,  $f(a)$  is in  $X_a$ .

Definition 5.2 For each  $a$  in the index set  $A$  define the function  $P_a$  with domain  $\prod \{X_a : a \in A\}$  and range  $X_a$  by

$$P_a(f) = f(a) \text{ for each } f \text{ in } \prod \{X_a : a \in A\}.$$

Each function  $P_a$  is called the projection map into the  $a$ -th coordinate space.

Definition 5.3 Let  $\{X_a : a \in A\}$  be a collection of topological spaces. A subbase for the product topology consists of all subsets of the product in the form  $P_a^{-1}[U_a]$  where  $U_a$  is an open set in  $X_a$  and  $a$  is in  $A$ . Consequently, a base for the product topology consists of all sets in the form  $\bigcap \{P_a^{-1}[U_a] : a \in F\}$  where  $F$  is a finite subset of the set  $A$ . Refer to

[3], Chapter 3 for more complete details.

Theorem 5.1 The projection map  $P_a$  on the product of topological spaces  $\prod \{X_a : a \in A\}$  is an open continuous map for each  $a$  in  $A$ .

Proof The continuity follows from the definition of  $P_a$ . Let  $W$  be an open set in the product topology, let  $a$  be a member of  $A$  and let  $f$  be in  $W$ . There is an element of the base in the form  $V = \bigcap \{P_b^{-1}[U_b] : b \in F\}$  containing  $f$  such that  $V \subset W$ . If  $a$  is in  $F$  then  $P_a[V] = U_a \subset P_a[W]$  where  $U_a$  is open in  $X_a$ . If  $a$  is not a member of  $F$  then  $P_a[V] = X_a$  and hence in both cases  $P_a(f)$  is an interior point of  $X_a$ . ■

Remark From this point forward the product  $\prod \{X_a : a \in A\}$  will be written simply as  $\prod X_a$  with the index set being understood as  $A$  unless stated specifically otherwise.

Theorem 5.2 Let  $\{X_a : a \in A\}$  be a collection of topological spaces. If each  $X_a$  is  $T_0$ ,  $T_1$ , or  $T_2$ , then the product  $\prod X_a$  is  $T_0$ ,  $T_1$ , or  $T_2$ , respectively.

Proof Suppose each space is  $T_2$ . Let the distinct functions  $f$  and  $g$  be members of  $\prod X_a$ . There is at least one  $a$  in  $A$  so that  $f(a) \neq g(a)$ . Since  $X_a$  is  $T_2$  there are disjoint open sets  $U$  and  $V$  in  $X_a$  such that  $U$  contains  $f(a)$  and  $V$  contains  $g(a)$ . The required disjoint open neighborhoods of  $f$  and  $g$  in  $\prod X_a$  are  $P_a^{-1}[U]$  and  $P_a^{-1}[V]$ . ■ (Similarly for  $T_0$ ,  $T_1$ .)

The next theorem is taken from [3].

Theorem 5.3 Let  $\{X_a : a \in A\}$  be a collection of topological spaces satisfying the first axiom of countability. The space  $\prod X_a$  is first countable if and only if all but a countable number of the  $X_a$ 's are indiscrete.

The proof is found in [3]. The next theorem appears as an exercise in [3].

Theorem 5.4 If  $\{X_a : a \in A\}$  is a family of topological spaces then  $\prod X_a$  is second countable if and only if each  $X_a$  is second countable and all but a countable number of the  $X_a$ 's are indiscrete.

Proof Suppose  $\prod X_a$  is a second countable. For each  $a$  in  $A$ ,  $G_a = \{P_a[B_k] : k = 1, 2, \dots\}$  is a countable base for  $X_a$ , where  $\{B_k\}$  is a countable base for  $\prod X_a$ . Now  $\prod X_a$  is first countable since it is second countable and thus by Theorem 5.3 all but a countable number of the spaces  $X_a$  are indiscrete.

Suppose each  $X_a$  is second countable and all but a countable number are indiscrete. Let  $D$  be the collection of all elements  $a$  of  $A$  such that  $X_a$  is not indiscrete. For each  $a$  in the countable set  $D$  let  $S_a$  be a countable base for  $X_a$ . The collection

$$\left\{ \bigcap \{P_a^{-1}[U_a] : a \in F\} : F \text{ is finite and } U_a \in S_a \right\}$$

where each  $F \subset A$ , is a countable base for  $\prod X_a$ . ■

The following theorem appears as an exercise in [3].

Theorem 5.5 Let  $\{X_a : a \in A\}$  be a collection of regular topological spaces. The product  $\prod X_a$  is regular.

Proof Let  $W$  be a neighborhood of the element  $f$  in the product space. Let  $B$  be an element of the base for the product topology in the form  $\bigcap \{P_a^{-1}[U_a] : a \in F\}$  where  $F$  is finite, which contains  $f$  and is contained in  $W$ . Each  $X_a$  is regular and hence there is a closed neighborhood  $V_a$  of  $f(a)$  contained in  $U_a$  for each  $a$  in  $F$ . Let  $V = \bigcap \{P_a^{-1}[V_a] : a \in F\}$ ,

then  $V$  is a closed neighborhood of  $f$  contained in  $W$ . ■

The next theorem appears as an exercise in [3].

Theorem 5.6 Let  $\{X_a : a \in A\}$  be a family of topological spaces. The product  $\prod X_a$  is connected if and only if each  $X_a$  is connected.

Proof Suppose each  $X_a$  is connected. Define the set  $A$ :

$$A = \left\{ g \in \prod X_a : f \text{ and } g \text{ are in some connected subset of } \prod X_a \right\},$$

where  $f$  is a fixed point of  $\prod X_a$ . Now to show that  $A$  is dense in the product. Let  $B$  be a member of the base of the product containing  $g$  in the form  $\bigcap \left\{ P_{a_k}^{-1}[U_{a_k}] : k=1, \dots, n \right\}$ , where each  $U_{a_k}$  is open in  $X_{a_k}$ . Define  $A_1$  by

$$A_1 = \left\{ h \in \prod X_a : h(a) = f(a) \text{ for each } a \text{ in } A - \{a_1\} \right\}.$$

For each  $k = 2, 3, \dots, n$  define the sets  $A_k$  by

$$A_k = \left\{ h \in \prod X_a : h(a_j) = g(a_j) \text{ for } j = 1, \dots, k-1 \right.$$

$$\left. \text{and } h(a) = f(a) \text{ for } a \neq a_1, \dots, a_k \right\}.$$

Each  $A_k$  is homeomorphic to  $X_{a_k}$  since  $P_{a_k}$  is a homeomorphism on  $A_k$  and therefore each  $A_k$  is connected. For  $k = 1, \dots, n$  let  $s_k$  be the element of the product such that  $s_k(a_j) = g(a_j)$  for  $j = 1, \dots, k$  and  $s_k(a) = f(a)$  otherwise. It follows that  $s_k \in A_{k+1}$  for  $k = 1, \dots, n-1$  and therefore

the union  $\bigcup_{k=1}^n A_k$  of connected sets is connected. Let  $t$  be the point of the product such that  $t(a_j) = g(a_j)$  for  $j = 1, \dots, n$ , and  $t(a) = f(a)$  otherwise. The point  $t$  is in  $A_n$  and hence in the union of the  $A_k$ 's, a connected set also containing  $f$  since  $f$  is in  $A_1$ . Hence  $t$  is a member of  $A$  contained in  $B$  and therefore, since  $g$  and  $B$  were chosen arbitrarily,  $A$  is dense in the product space. The product space is the closure of the connected set  $A$  and is therefore connected.

Suppose  $\prod X_a$  is connected. Each  $X_a$  being the continuous image of a connected space is connected. ■

Theorem 5.7 Let  $\prod X_a$  be a product of topological spaces. If  $\prod X_a$  is either  $T_0$ ,  $T_1$ , or  $T_2$ , then so is each  $X_a$ .

Proof Suppose  $\prod X_a$  is  $T_2$ . Let  $b$  be in  $A$  and let  $x$  and  $y$  be distinct points in  $X_b$ . Let  $f$  be any member of the product with the property that  $f(b) = x$ . Define the member  $g$  of the product space by  $g(a) = f(a)$  for each  $a$  in  $A$  such that  $a \neq b$  and  $g(b) = y$ . Consequently  $f \neq g$  and therefore there are disjoint neighborhoods  $U^*$  and  $V^*$  of  $f$  and  $g$  respectively. Let  $U$  and  $V$  be base elements containing  $f$  and  $g$  respectively which are contained in  $U^*$  and  $V^*$  respectively. The sets  $U$  and  $V$  will be of the form  $U = \bigcap \{P_a^{-1}[U_a] : a \in F_1\}$ , and  $V = \bigcap \{P_a^{-1}[V_a] : a \in F_2\}$ , where each  $U_a$  is a neighborhood of  $f(a)$  and each  $V_a$  is a neighborhood of  $g(a)$ . If  $b$  is not in  $F_1$  then  $g$  is in  $U$ , a contradiction, and if  $b$  is not in  $F_2$  then  $f$  is in  $V$ , a contradiction. Hence  $b$  is in  $F_1 F_2$ . Suppose there is a  $z$  in  $U_b V_b$ , then define  $h$  by  $h(a) = f(a)$  for  $a \neq b$  and  $h(b) = z$ . It follows that  $h$  is in both  $U$  and  $V$ , a contradiction. Hence  $U_b$  and  $V_b$  are disjoint neighborhoods of  $x$  and  $y$  respectively. ■

The above theorem serves as a converse to Theorem 5.2. The next

theorem is the converse to Theorem 5.3.

Theorem 5.8 If the product of topological spaces  $\prod X_a$  is first countable, then so is each  $X_a$ .

The proof follows from the open continuous property of the projection mappings.

Theorems 5.9 - 5.12 are taken from [3].

Theorem 5.9 The product of Tychonoff spaces is a Tychonoff space.

Theorem 5.10 (Tychonoff) The product of compact spaces is compact.

Theorem 5.11 If the product of a collection of topological spaces is locally compact then each coordinate space is locally compact and all but a finite number are compact.

Theorem 5.12 The product of uniform spaces is complete if and only if each coordinate space is complete.

The product of normal spaces need not be normal. For an example see [3], Problem E, p. 131.

Let  $\{X_a : a \in A\}$  be a collection of topological spaces. For each  $a$  in  $A$  let  $Y_a$  be a subset of  $X_a$ . There are two ways in which a topology on the product  $\prod Y_a$  can be imposed. First, each  $Y_a$  can be considered a topological space (with the subspace topology) and hence the product would have the product topology. Secondly,  $\prod Y_a$  can be considered a subset of the product  $\prod X_a$  and thus would have the relative topology in  $\prod X_a$ . It is an easy exercise to show that both of these topologies are equivalent.



## CHAPTER VI

## TOPOLOGICAL GROUPS

If a topological space  $X$  is also a group in which the group operations are "continuous" then  $X$  has special inheritance properties which will be investigated in this chapter. The main reference for this chapter is [5]. The group operations will be written using multiplicative notation and the intersections of sets will be indicated by  $\cap$  in order to distinguish them from products of subsets of a group.

Definition 6.1 A set  $G$  of elements is called a topological group if and only if (1)  $G$  is a group, (2)  $G$  is a topological space, and (3) for each pair of elements  $a$  and  $b$  of  $G$ , and each neighborhood  $W$  of  $ab^{-1}$ , there are neighborhoods  $U$  and  $V$  of  $a$  and  $b$  respectively such that  $UV^{-1} \subset W$ .

Lemma Let  $G$  be a topological group and let  $a$  be an element of  $G$ . The function defined by  $f(x) = ax$  for each  $x$  in  $G$  is a homeomorphism of  $G$  onto  $G$ .

Proof Clearly  $f$  is 1-1. The continuity properties follow from (3) of Definition 6.1. ■

Lemma Let  $G$  be a topological group and let  $U$  be an open subset of  $G$ . If  $a \in G$  then  $aU$  and  $Ua$  are open sets, consequently,  $VU$  and  $UV$  are open sets for each subset  $V$  of  $G$ .

Proof The set  $aU$  is the homeomorphic image of  $U$ .

Theorem 6.1 Every topological group  $G$  is regular.

Proof Let  $U$  be a neighborhood of the identity  $e$ . Since  $e = ee^{-1}$

there are neighborhoods  $V_1$  of  $e$  and  $V_2$  of  $e^{-1}$  such that  $V_1 V_2^{-1} \subset U$ . Let  $p$  be a member of the closure  $\bar{V}$ , where  $V = V_1 \cap V_2$ . Now  $eeV V^{-1} \subset V_1 V_2^{-1} \subset U$  and hence  $pV$  is a neighborhood of  $p$ . There is a  $y$  in  $V$  so that  $py = zeV$ , since  $p$  is a point of  $V$  or a limit point of  $V$ . It follows that  $p = zy^{-1}e \in VV^{-1} \subset U$  and thus  $\bar{V} \subset U$ . If, on the other hand,  $U$  is a neighborhood of the point  $x$  in  $G$ , then apply the preceding argument to the neighborhood  $x^{-1}U$  of  $e$  to obtain the neighborhood  $V$  such that  $eeV \subset \bar{V} \subset x^{-1}U$ . The required closed neighborhood of  $x$  is  $x\bar{V}$ .

In the preceding theorem it is shown that if  $G$  possesses the property of being regular locally at  $e$ , then  $G$  inherits the property. The next theorem is a generalization of one found in [5], in which second countability is assumed. First, however, some results on nets are needed.

The net  $S$  with domain  $D$  will be written  $\{S_n, n \in D\}$ . The symbol  $S_n$  will be used in place of  $S(n)$  for each  $n$  in  $D$ . If  $\{S_n, n \in D\}$  is a net, then the net  $\{T_k, k \in E\}$  is a subnet of  $S$  if and only if there is a function  $n$  on  $E$  with range in  $D$  so that  $T(k) = S(n(k))$  for each  $k$  in  $E$ , and for each member  $m$  of  $D$  there is a member  $p$  of  $E$  so that  $k \succeq p$  implies  $n(k) \geq m$ , where  $\succeq$  is the relation that directs the set  $E$  and  $\geq$  directs  $D$ . The subnet  $T$  will be written  $\{S_{n_k}, k \in E\}$ .

Lemma. A subset  $A$  of a topological space is compact if and only if each net whose range in  $A$  has a subnet which converges to some point of  $A$ . See [3].

Theorem 6.2 If  $P$  and  $Q$  are compact subsets of a topological group  $G$ , then the product  $PQ$  is compact.

Proof Let  $\{S_n, n \in D\}$  be a net in  $PQ$ . For each  $n$  in  $D$  there is an  $R_n$  in  $P$  and a  $T_n$  in  $Q$  such that  $S_n = R_n T_n$ . Thus  $\{R_n, n \in D\}$  and  $\{T_n, n \in D\}$

are nets in  $P$  and  $Q$  respectively. Since  $P$  is compact, there is a subnet  $\{R_{n_k}, k \in E\}$  of  $R$  which converges to a point  $r$  in  $P$ . Now  $\{T_{n_k}, k \in E\}$  is a net in the compact space  $Q$  and hence there is a subnet  $\{T_{n_{k_j}}, j \in F\}$  which converges to a point  $t$  in  $Q$ . Let  $W$  be a neighborhood of  $rt$ . There are neighborhoods  $U$  and  $V$  of  $r$  and  $t$  respectively, such that  $UV \subset W$ . There are elements  $N_1$  and  $N_2$  in the set  $F$  so that if  $j \geq N_1$  then  $R_{n_{k_j}} \in U$  and if  $j \geq N_2$  then  $T_{n_{k_j}} \in V$ . There is an  $N$  in  $F$  such that  $N \geq N_1$  and  $N \geq N_2$ . If  $j \geq N$  then  $R_{n_{k_j}} T_{n_{k_j}} = S_{n_{k_j}} \in UV \subset W$ . Therefore  $\{S_{n_{k_j}}, j \in F\}$  converges to the element  $rt$  in  $PQ$ . ■

Lemma Let  $G$  be a topological group,  $S$  a basis for the neighborhood system of the identity  $e$ , and  $M$  a set dense in  $G$ . Then  $B = \{Ux : U \in S \text{ and } x \in M\}$  is a base for  $G$  and

- (1) If  $U, V \in S$  then there is a  $W \in S$  such that  $W \subset U \cap V$ ;
- (2) If  $U \in S$  then there is a  $V$  in  $S$  such that  $VV^{-1} \subset U$ ;
- (3) If  $U \in S, a \in U$ , there is a  $V \in S$  such that  $Va \subset U$ ;
- (4) If  $U \in S, a \in G$  then there is a  $V \in S$  such that  $a^{-1}Va \subset U$ .

For a proof see [5].

It will now be shown that the topology for a topological group is inherited from the neighborhood system of the identity.

Theorem 6.3 Let  $G$  be a group and let  $S$  be a collection of subsets of  $G$  which satisfy conditions (1) - (4) of the preceding lemma. A topology for  $G$  is uniquely determined such that  $G$  is a topological group and  $S$  is a base for the neighborhood system of the identity  $e$ .

Proof Let  $B = \{Ux : U \in S \text{ and } x \in G\}$ . Let  $a \in Ux \cap Vy$ . Thus  $ax^{-1} \in U$  and hence by (3) there is a  $W$  in  $S$  so that  $Wax^{-1} \subset U$ . Then  $Wa \subset Ux$

and  $ay^{-1} \in V$  so that there is a  $N$  in  $S$  with the property that  $Nay^{-1} \subset V$  and hence  $Na \subset Vy$ . It follows then that  $(N \cap W)a \subset Ux \cap Vy$  and therefore, by (1), there is an  $O$  in  $S$  such that  $O \subset N \cap W$ . Hence  $Oa \subset Ux \cap Vy$  and since  $e \in O$ ,  $a \in Oa$ . Hence  $B$  is a basis for a topology for  $G$ .

Let  $x \in G$ ,  $y \in G$ ,  $N \in S$ ,  $b \in G$ , and  $xy^{-1} \in Nb$ . There is a  $W$  so that  $Wxy^{-1} \subset Nb$ , and  $a \in S$  so that  $UU^{-1} \subset W$  and a  $V$  in  $S$  so that  $xy^{-1}Vyx^{-1} \subset U$ . Therefore  $xy^{-1}V^{-1} \subset U^{-1}xy^{-1}$  and  $Ux(Vy)^{-1} = Uxy^{-1}V^{-1} \subset UU^{-1}xy^{-1} \subset Wxy^{-1} \subset Nb$ . Hence (3) of Definition 6.1 is satisfied and  $G$  is a topological group with base  $B$ .

Let  $W$  be an open set containing  $e$ . There is a neighborhood  $Ua$  of  $e$  such that  $Ua \subset W$ . Since  $ee \in Ua$  then  $a^{-1} \in U$ . Hence there is a  $V$  in  $S$  such that  $Va^{-1} \subset U$  or  $V \subset Ua \subset W$ . Thus  $V \in S$  and  $ee \in V \subset W$ . Hence  $S$  is a base for the neighborhood system of  $e$ .

In order to show uniqueness of the topology, let  $T$  be a topology for  $G$  such that  $S$  is a basis for the neighborhood system of  $e$  and  $G$  is a topological group. Both  $S$  and  $B$  are subsets of  $T$ . Let  $W \in T$  and let  $a \in W$ . Then  $ee \in Wa^{-1} \in T$ . There is a  $U$  in  $S$  such that  $ee \in U \subset Wa^{-1}$  and hence  $a \in Ua \subset W$ . Therefore  $B$  is a basis for  $T$ . ■

Definition 6.2 Let  $G$  be a topological group and let  $H$  be one of its subgroups. Let  $G/H$  denote the set of all right cosets of  $H$  in  $G$ . Let the topology for  $G/H$  be the topology of the decomposition space  $G/D$  with the decomposition  $D$  given by  $D = \{Hx: x \in G\}$ ,  $G/H$  is called the quotient group of  $G$ .

Definition 6.3 Let  $G$  be a topological group. A subgroup of the group  $G$ ,  $H$ , will be called a subgroup of the topological group  $G$  if and only if  $H$  is closed in the topology of  $G$ .

Theorem 6.4 Let  $G$  be a topological group. Each subgroup of  $G$  is a topological group (but may not be a subgroup of the topological group).

Proof Let  $A$  be a subgroup of  $G$ . Let  $a, b \in A$  and  $W = W_1 \cap A$  where  $W_1$  is open in  $G$  and  $ab^{-1} \in W_1$ . There are neighborhoods  $U_1$  and  $V_1$  of  $a, b$  respectively in  $G$  such that  $U_1 V_1^{-1} \subset W_1$ . Let  $U = A \cap U_1$  and  $V = A \cap V_1$  then  $U \subset U_1$  and  $U \subset A$ . Thus  $V^{-1} \subset V_1^{-1}$  and  $V^{-1} \subset A$ . Therefore  $UV^{-1} \subset W_1 A = W$  and hence (3) of Definition 6.1 is satisfied. ■

Remark If  $A$  is a subgroup of the topological group  $G$  then  $A$  is closed by definition and from previous chapters it follows that compactness and local compactness are inherited.

Theorem 6.5 Let  $G$  be a topological group and let  $H$  be a subgroup of  $G$ . The projection map  $P$  into  $G/H$  is an open map.

Proof Let  $U$  be an open set in  $G$ . Now  $P^{-1}P[U] = HU$  and is therefore open in  $G$ . By the definition of quotient topology on  $G/H$ ,  $P[U]$  is open in  $G/H$ . ■

Theorem 6.6 If  $G$  is a topological group and  $H$  is a normal subgroup of  $G$  then the projection map  $P$  is a homomorphism on  $G$  onto  $G/H$  and for each  $x \in G$ ,  $P(x) = Hx$ .

Theorem 6.7 If  $H$  is a normal subgroup of topological group  $G$  then  $G/H$  is a topological group.

Proof Let  $W$  be an open set in  $G/H$  such that  $(Hx)(Hy)^{-1} \in W$ . The set  $B = P^{-1}[W]$  is open in  $G$ . Now  $(Hx)(Hy)^{-1} = Hxy^{-1} \in W$ . Hence there is a  $t$  in  $B$  so that  $Hxy^{-1} = P(t) = Ht$  and therefore  $xy^{-1} = th$  for some  $h$  in  $H$ . In other words  $x(hy)^{-1} = t \in B$ . Since  $G$  is a topological group there is an open set  $U$  containing  $x$  and an open set  $V$  containing  $hy$  such that  $UV^{-1} \subset B$ . Observe that  $P[UV^{-1}] = P[U](P[V])^{-1}$  which is a subset of  $W$ , where  $P[U]$  and  $P[V]$  are open sets containing  $Hx$  and  $Hy$  respectively. ■

Theorem 6.8 Let  $G$  be a topological group and let  $H$  be a normal

subgroup of  $G$ . If  $G$  is second countable then  $G/H$  is second countable.

Proof If  $B = \{B_n\}$  is a countable base for  $G$  then, since  $P$  is open,  $\{P[B_n]\}$  is a countable base for  $G/H$ . ■

Theorem 6.9 If  $G$  is a compact (locally compact) topological group then each of its quotient groups is compact (locally compact).

Proof The assertion follows from the open continuous property of the projection map  $P$ .

Theorem 6.10 Let  $G$  be a topological group which is second countable. If  $H$  is a compact normal subgroup of  $G$  and if  $G/H$  is compact then  $G$  is compact.

Proof Let  $\{a_n\}$  be a sequence of elements of  $G$ . Then  $\{P(a_n)\}$  has a convergent subsequence  $\{P(a_{n_k})\}$  which converges to some  $a^*$  in the compact  $G/H$ . Let  $\{B_k\}$  be a basis of the neighborhood system of the identity  $e$  in  $G$  such that each  $B_k \supset B_{k+1}$ . Now  $P[B_k]$  is a neighborhood system of the identity  $H$  in  $G/H$  and  $P[B_k] \supset P[B_{k+1}]$  for each  $k = 1, \dots$ . Since  $P(a_{n_k})$  converges to  $a^*$  then  $P(a_{n_k})a^{*-1}$  converges to  $H$ . For each positive integer  $m$ , choose an integer  $n$  such that  $P(a_{n_k})a^{*-1} \in P(B_m)$  and so that a subsequence of the  $a_{n_k}$ 's is formed with  $x_m = a_{n_k}$ . Now  $\{x_m\}$  is a subsequence of  $\{a_{n_k}\}$  such that  $P(x_m)a^{*-1} \in P(B_m)$  for each positive integer  $m$ . Let  $a \in P^{-1}(a^*)$  and for each positive integer  $m$  choose a  $b_m \in (P(x_m)a^{*-1}) \cap B_m$ . Let  $c_n = b_n a$  for each positive integer  $n$ . Then  $\{c_n\}$  converges to  $a$ . Now  $P(c_n) = P(b_n)P(a) = P(x_n)a^{*-1}a^* = P(x_n)$ . Let  $d_n = x_n c_n^{-1}$ . Then the sequence  $\{d_n\}$  is a subset of the compact set  $H$  and hence there is a subsequence  $\{d_{k_n}\}$  which converges to some  $d$  in  $H$ . Since  $\{c_{k_n}\}$  converges to  $a$  and  $\{d_{k_n}\}$  converges to  $d$  it follows that  $\{d_{k_n} c_{k_n}\}$  converges and hence

$\{x_{k_n}\}$  converges which in turn is a subsequence of  $\{a_n\}$  which converges. ■

Theorem 6.11 Let  $G$  be a topological group. If  $H$  is a subgroup of  $G$  then  $\overline{H}$  is a subgroup of the topological group  $G$ . If  $H$  is a normal subgroup then so is  $\overline{H}$ .

Proof Suppose  $H$  is a subgroup of  $G$ . Let the elements  $a$  and  $b$  be members of  $\overline{H}$ . Let  $W$  be a neighborhood of  $ab^{-1}$ . There are neighborhoods  $U$  and  $V$  of  $a$  and  $b$  respectively such that  $UV^{-1} \subset W$ . There are points  $x$  and  $y$  of  $H$  such that  $x$  is in  $U$  and  $y$  is in  $V$ . Now  $xy^{-1}$  is a member of  $H$  such that  $xy^{-1} \in UV^{-1} \subset W$  and therefore  $ab^{-1} \in \overline{H}$ . Thus  $H$  forms a group.

Suppose  $H$  is normal. Let  $h$  be in  $\overline{H}$  and let  $a$  be in  $G$ . Let  $W$  be a neighborhood of  $aha^{-1}$ . Since  $a^{-1}Wa$  is a neighborhood of  $h$  there is an element  $h_1$  of  $H$  so that  $h_1 \in a^{-1}Wa$  or in other words  $ah_1a^{-1} \in W$ . Since  $H$  is a normal subgroup it follows that  $ah_1a^{-1} \in H$  and hence  $aha^{-1} \in \overline{H}$ . ■

Definition 6.4 Let  $G$  and  $G^*$  be topological groups. A function  $f$  on  $G$  onto  $G^*$  is an isomorphism on the topological group  $G$  onto the topological group  $G^*$  if and only if  $f$  is an isomorphism on the group  $G$  onto the group  $G^*$  and  $f$  is a homeomorphism on the topological space  $G$  onto the topological space  $G^*$ .

Definition 6.5 A function  $g$  on a topological group  $G$  into a topological group  $G^*$  is a homomorphism on the topological group  $G$  into the topological group  $G^*$  if and only if  $g$  is a homomorphism on the group  $G$  into the group  $G^*$  and  $g$  is a continuous function on the topological space  $G$  into the topological space  $G^*$ .

Remark Let  $G$  be a topological group and let  $H$  be a normal subgroup of  $G$ . The projection map  $P$  is a homomorphism on the topological group  $G$  onto

the topological group  $G/H$ .

Theorem 6.12 Let  $f$  be a homomorphism on the group  $G$  into the group  $G^*$ . (1) If for each neighborhood  $U^*$  of the identity  $e^*$  in  $G^*$  there is a neighborhood  $U$  of the identity  $e$  in  $G$  such that  $f[U] \subset U^*$ , then  $f$  is continuous on  $G$ . (2) If for each neighborhood  $U$  of  $e$  there is a neighborhood  $U^*$  of  $e^*$  such that  $U^* \subset f[U]$ , then  $f$  is open on  $G$ .

Proof Suppose the condition in (1) is satisfied. Let the point  $a$  be a member of  $G$  and let  $U^*$  be a neighborhood of  $f(a)$  in  $G^*$ . The set  $U^*f(a)^{-1}$  is a neighborhood of  $e^*$ . Hence there is a neighborhood  $V$  of  $e$  such that  $f[V] \subset U^*f(a)^{-1}$ . It follows that  $f(Va) = f(V)f(a) \subset U^*$ .

Suppose the condition in (2) is satisfied. Let  $U$  be an open set in  $G$ ; let  $a \in U$ . Now  $Ua^{-1}$  is a neighborhood of  $e$  and hence there is a  $U^*$ , open in  $G^*$ , which contains  $e^*$  such that  $U^* \subset f[Ua^{-1}]$ . Thus  $U^*f(a) \subset f[Ua^{-1}]f(a) = f[U]$  and hence  $U^*f(a)$  is the desired neighborhood of  $f(a)$ . ■

Theorem 6.12 could be rephrased as follows. If the homomorphism  $f$  on the group  $G$  into the group  $G^*$  is continuous at the identity  $e$  then  $f$  is continuous everywhere on  $G$ . If  $f$  is open at the identity  $e$  then  $f$  is open on  $G$ .

Lemma Let  $\{H_a : a \in A\}$  be a collection of subgroups of a topological group  $G$ . The intersection  $\bigcap \{H_a : a \in A\}$  is a subgroup of the topological group  $G$ .

Theorem 6.13 Let  $H$  be a subgroup of the topological group  $G$  and let  $N$  be a normal subgroup of the topological group  $G$ . The intersection  $H \cap N$  is a normal subgroup of the topological group  $H$ .

Proof The fact that  $H \cap N$  is a normal subgroup of  $H$  is an elementary result in group theory. The subgroup, being the intersection of two



closed sets, is closed. ■

Theorem 6.1<sup>1</sup> Let  $H$  and  $N$  be subgroups of a topological group  $G$ .

If either  $H$  or  $N$  is compact then the product  $HN$  is a topologically closed set in  $G$ .

Proof Suppose that  $H$  is compact. Let  $\{S_n, n \in D\}$  be a net in  $HN$  which converges to a point  $c$  in  $G$ . Then there are elements  $R_n$  and  $T_n$  in  $H$  and  $N$  respectively for each  $n$  in  $D$  such that  $S_n = R_n T_n$ . Thus  $\{R_n, n \in D\}$  and  $\{T_n, n \in D\}$  are nets in  $H$  and  $N$  respectively. Since  $H$  is compact, there is a subnet  $\{R_{n_k}, k \in E\}$  such that it converges to a point  $a$  in  $H$ . But  $\{S_{n_k}, k \in E\}$  converges to  $c$  and hence  $\{T_{n_k}, k \in E\}$  converges to  $a^{-1}c$ . Since  $N$  is closed it follows that  $a^{-1}c \in N$  and therefore  $c$  is a member of  $HN$ . ■

The preceding theorem appears in [5] with the additional assumption of second countability. The use of nets in place of subsequences allows second countability to be dropped.

Definition 6.6 Let  $A$  be a subset of a group  $G$ . Let the symbol  $G[A]$  denote the group formed by the intersection of the family of subgroups of  $G$  which contain  $A$ . The group  $G[A]$  is, then, the minimal group of  $G$  which contains  $A$ .

Definition 6.7 Let  $\{G_a : a \in A\}$  be a collection of subgroups of a group  $G$ . The group  $G$  is the internal direct product,  $\bigotimes \{G_a : a \in A\}$ , of the elements of  $\{G_a : a \in A\}$  if and only if

(1)  $G_a$  is a normal subgroup for each  $a \in A$ :

(2)  $G[\bigcup_{a \in A} G_a] = G$

(3)  $G_b \cap G[\bigcup_{a \in A} G_a - G_b] = \{e\}$ .

As with the Cartesian product, the internal direct product will be abbreviated  $\bigotimes G_a$  with the index set being understood to be  $A$ .

Definition 6.8 Let  $\{G_a : a \in A\}$  be a collection of subgroups of a group  $G$ . The Cartesian product  $\prod G_a$  forms a group under the operation  $*$  defined by: let  $f$  and  $g$  be in  $\prod G_a$ , then  $f * g$  is the function defined by  $(f * g)(a) = f(a)g(a)$  for each  $a$  in  $A$ .

Lemma Let  $\{G_a : a \in A\}$  be a family of subgroups of a group  $G$ . The projection map  $P_a$  is a homomorphism on the group  $\prod G_a$  onto the group  $G_a$  for each  $a$  in  $A$ .

Theorem 6.15 If  $\{G_a : a \in A\}$  is a family of topological groups, then  $\prod G_a$  is a topological group.

Proof Only (3) of Definition 6.1 needs to be verified. Let  $f$  and  $g$  be members of  $\prod G_a$  and let  $W^*$  be a neighborhood of  $fg^{-1}$  (where  $^{-1}$  denotes the inverse element in the group and not the function inverse). There is an element  $W$  of the base in the form  $\bigcap \{P_a^{-1}[B_a] : a \in I\}$  such that  $W \subset W^*$ , where  $I$  is a finite subset of  $A$  and for each  $a$  in  $I$ ,  $B_a$  is open in  $G_a$ , and  $f(a)(g(a))^{-1} \in B_a$ . Since each  $G_a$  is a topological group there are open sets  $U_a$  and  $V_a$  in  $G_a$  such that  $f(a) \in U_a$ ,  $g(a) \in V_a$ , and  $U_a(V_a)^{-1} \subset B_a$ . Let  $U = \bigcap \{P_a^{-1}[U_a] : a \in I\}$  and  $V = \bigcap \{P_a^{-1}[V_a] : a \in I\}$ . Thus  $UV^{-1}$  is the required neighborhood of  $fg^{-1}$  which is contained in  $W$ . ■

Definition 6.9 Let  $\{G_a : a \in A\}$  be a family of groups. The internal Cartesian product,  $\prod 'G_a$ , is the subgroup of  $\prod G_a$  consisting of the elements  $f$  of  $\prod G_a$  such that  $f(a)$  is the identity in  $G_a$  for all except a finite number of  $a$ 's in  $A$ .

Theorem 6.16 Let  $\{G_a : a \in A\}$  be a family of subgroups of a group  $G$ , where  $G$  is a topological group. If  $G = \bigotimes G_a$  then the topological group

$G$  is isomorphic to the topological group  $\prod G_a$ .

Proof It is known, from group theory, that the function  $h$  is an isomorphism on  $\prod G_a$  onto  $G$  where  $h$  is defined by  $h(f) = \times f(a)$ , the product of the  $f(a)$ 's in  $G$  for each  $a$  in  $A$ .

Let  $f \in \prod G_a$ . There is a finite set  $B$  such that for each  $a \in A - B$ ,  $f(a)$  is the identity. By induction, for each  $b$  in  $B$  there is an open set  $U_b^*$  in  $G$  so that  $f(b) \in U_b^*$  and  $\times U_b^* \subset W$ . Define  $U_b = G_b \cap U_b^*$  for each  $b$  in  $B$ . Let the set  $V$  be  $\{p_b^{-1}[U_b] : b \in B\}$ . Clearly  $V$  is a neighborhood of  $f$  in  $\prod G_a$  such that  $h[V] \subset W$  and hence  $h$  is continuous. Let  $W$  be an open set in  $\prod G_a$ . Let  $x$  be in  $h[W]$ . There is an  $f$  in  $W$  so that  $x = h(f) = \times f(b)$ . There is a base element  $U$  in the form  $\bigcap \{p^{-1}[U_a] : a \in B\}$  such that  $f \in U \subset W$ . Now  $h[U] = \times U_a$  and is hence an open set of  $h[W]$  containing  $x$ . Thus  $h$  is 1-1, continuous, and open and therefore a homeomorphism. ■

## INDEX

	Page
base for a uniformity . . . . .	13
Cartesian product of sets . . . . .	38
Cartesian product of groups . . . . .	53
Cauchy net. . . . .	19
closed functions. . . . .	21
compact . . . . .	8, 29, 43, 48
compact functions . . . . .	22
compact subsets of a topological group. . . . .	45
complete space. . . . .	20, 43
completely normal . . . . .	7
connected . . . . .	5, 29
connected im kleinen. . . . .	8
continuous functions. . . . .	21, 51
countably compact . . . . .	10
cover . . . . .	17
decomposition . . . . .	27
decomposition space . . . . .	27
decomposition for subspaces . . . . .	29
directed system . . . . .	18
discrete families . . . . .	11
even covers . . . . .	18
first countability. . . . .	4, 39, 43

	Page
Hausdorff . . . . .	6, 31
homeomorphism . . . . .	21, 44
homomorphism. . . . .	48
internal Cartesian product. . . . .	53
internal direct product . . . . .	52
isomorphism . . . . .	50
light functions . . . . .	25
Lindelöf space . . . . .	29
locally compact . . . . .	8, 36, 43, 48
locally connected . . . . .	8, 36
locally finite families . . . . .	10
lower semi-continuous decompositions. . . . .	30
metrizability . . . . .	17
monotone functions. . . . .	24
monotone-light decompositions . . . . .	37
neighborhood system . . . . .	46
net . . . . .	19
normal subspaces. . . . .	6, 12
normal subgroups. . . . .	50
open functions. . . . .	21, 51
paracompact . . . . .	12
product topology. . . . .	38
projection map. . . . .	27, 38, 48, 50
pseudo-metric . . . . .	16
pseudo-metrizability. . . . .	16
quasi-compact functions . . . . .	23
quasi-monotone functions. . . . .	24

	Page
quasi-open functions. . . . .	24
quotient group. . . . .	47
quotient topology . . . . .	27
refinement of a cover . . . . .	12
regular . . . . .	7, 32, 40, 44
retraction. . . . .	26
second countability . . . . .	4, 32, 49
separable . . . . .	5
separable metric spaces . . . . .	33, 36
separated sets. . . . .	5
sequentially compact. . . . .	10
sigma-discrete families . . . . .	11
sigma-locally finite families . . . . .	11
subgroup of a topological group . . . . .	47
$T_0$ , $T_1$ , $T_2$ . . . . .	6, 39
topological groups. . . . .	44
topological space . . . . .	3
totally bounded . . . . .	20
totally disconnected. . . . .	5
Tychonoff space . . . . .	7, 43
Tychonoff theorem . . . . .	43
uniform continuity. . . . .	15
uniform cover . . . . .	17
uniform topological space . . . . .	14
uniformity. . . . .	13
upper semi-continuous decompositions. . . . .	29, 33
Urysohn's metrization theorem . . . . .	33

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